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PROCEEDINGS

OF THE

LONDON MATHEMATICAL SOCIETY.

VOL. VI.

FROM NOVEMBER, 1874, TO NOVEMBER, 1875.

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PROCEEDINGS

OF THE

LONDON MATHEMATICAL SOCIETY.

VOL. VI.

ELEVENTH SESSION, 1874–5.

November 12th, 1874.

ANNUAL GENERAL MEETING, held at 22, Albemarle Street,

Dr. HIRST, F.R.S., President, in the Chair.

THE Chairman, in feeling terms, informed the Meeting of the death of Dr. Otto Hesse, of Munich, who had been elected an Honorary Foreign Member, Dec. 14th, 1871.

The Treasurer (S. Roberts, Esq., M.A.) read his Report, which was accepted; and Mr. J. Stirling was requested to act as Auditor for the same.

The Report of the Secretaries was read, and on the motion of Mr. J. Glaisher, F.R.S., seconded by Mr. Finlaison, was adopted. From this Report it appeared that the number of Members had increased from 113 to 118 during the Session.

The Obituary of the Society, during the Session, contains the name of one Foreign Member, Dr. Otto Hesse, of the Royal Polytechnicum, Munich.

The communications made to the Society during the past Session had been as follows:—

"Description of a new Instrument for converting Circular into General Rectilinear Motion, and into Motion in Conics and Higher Plane Curves:" by Mr. J. J. Sylvester, F.R.S.

"The expression of the Arc of a Cartesian by Elliptic Functions:" by Mr. Samuel Roberts, M.A.

"Graphic representation of the Harmonic Components of a Periodic Motion:" by Prof. W. K. Clifford, F.R.S.

"Steiner's Surface:" by Prof. Cayley, F.R.S.

"On the Transformation of Continued Products into Continued Fractions:" by Mr. J. W. L. Glaisher, M.A.

"The Foundations of the Differential Calculus and of Dynamics:" by Prof. W. K. Clifford, F.R.S.

"A Method of treating the Kinematical Question of the most general Displacement of a Solid in Space:" by Prof. Crofton, F.R.S.

"On Hamilton's Characteristic Function for a Narrow Beam of Light:" by Prof. Clerk-Maxwell, F.R.S.

"The Free Motion of a Solid in Elliptic Space:" by Prof. Clifford, F.R.S.

"Note on the Inversion of Bernoulli's Theorem in Probabilities:" by Mr. C. J. Monro, B.A.

"On certain constructions for Bicircular Quartics;" and "On a Geometrical Interpretation of the Equations obtained by equating to zero the Resultant and the Discriminant of two Binary Quantics:" by Prof. Cayley, F.R.S.

"On the Cartesian Equation of the Circle which cuts three given Circles at given Angles:" by Mr. J. Griffiths, M.A.

"On another System of Poristic Equations:" by Prof. Wolstenholme, M.A.

"On Probable Error in Statistics;" and a Note "On the Combination of Statistics:" by Mr. G. H. Darwin, M.A.

"Determination of the form of the Dome of Uniform Stress:" by Mr. C. W. Merrifield, F.R.S.

"Algebra considered as the Calculus of Similar Triangles:" by Mr. A. J. Ellis, F.R.S.

"Note on the Higher Singularities of Plane Curves:" by Prof. H. J. S. Smith, F.R.S.

"Inversion: with special reference to the Inversion of an Anchor Ring:" by Mr. H. M. Taylor, M.A.

"The Correlation of Two Planes:" by Dr. T. A. Hirst, F.R.S., President.

"The Contact of Quadrics with other Surfaces:" by Mr. W. Spottiswoode, F.R.S.

"A Rotating Sphere filled with Viscous Fluid:" by Mr. J. H. Röhrs, M.A., communicated by Prof. Cayley, F.R.S.

"The Parallels of Developables, and of Curves of Double Curvature:" by Mr. Samuel Roberts, M.A., Treasurer.

"Note on the Numerical Calculation of the Roots of Fluctuating Functions:" by Lord Rayleigh, F.R.S.

"On a remarkable relation between the difference of two Fagnanian

Arcs of an Ellipse of eccentricity e and that of two Corresponding Arcs of a Hyperbola of eccentricity $\frac{1}{e}$:" by Mr. J. Griffiths, M.A.

"On Stability of a Dynamical System, with two Independent Motions;" "On Rocking-Stones;" and "Small Oscillations to any degree of Approximation:" by Mr. E. J. Routh, F.R.S.

Shorter communications were also made by Mr. W. Marsham Adams, Lord Rayleigh, Prof. Cayley, Mr. Samuel Roberts, and Mr. Perigal.

The same Mathematical Journals had been subscribed for as in the preceding Session, with the exception of the "Mathematical Reprint from the 'Educational Times.'" This, as well as the 'Educational Times' itself, is now presented to the Society by the publishers.

The proceedings had been sent to the Physical and Medical Society of Erlangen, in addition to the persons and Societies named in the last Report.

The Society had received the proceedings of the above-named Society, in addition to those mentioned in a previous Report.

The following presents had been made to the Society:—

"Wilson's Elementary Geometry," third edition, 1873.

"Note sur la Différentiation et l'Intégration d'une Intégrale multiple par rapport à une Constante," par D. Bierens de Haan.

"La Méthode d'Euler pour l'Intégration de quelques équations différentielles linéaires démontrée à l'aide de l'équation intégrante," par D. Bierens de Haan.

"The Problem of Pythagoras," by W. Marsham Adams.

"Handbooks to the Mensurator and Cælometer," by the same.

"Ueber die verschiedenen Sturmschen Reihen und ihre gegenseitigen Beziehungen," von L. Kronecker.

"Ueber Deformationen elastischer isotroper Körper durch mechanische an ihrer Oberfläche wirkende Kräfte," von C. W. Borchhardt.

"Alfred Clebsch—Versuch einer Darlegung und Würdigung seiner wissenschaftlichen Leistungen von einigen seiner Freunde," Leipzig, 1873.

"Nautical Almanac" for 1877.

"Special Report on Immigration, accompanying information for immigrants relative to the prices and rentals of land, the staple products, facilities of access to markets, &c., to which are appended Tables, &c., by Edward Young, Ph.D., Chief of the Bureau of Statistics," Washington, 1872.

"Jahrbuch über die Fortschritte der Mathematik," von Carl Ohrtmann. "Felix Müller, Albert Wangerin," dritter Band, Jahrgang 1871, in 3 Heften, and vierter Band, Heft 1, Jahrgang 1872.

"Geometric Maps, exhibiting the method of delineating Curves through the Intersections of Trigonometrical Lines," by H. Perigal, F.R.A.S.

"Pfaff's Disquisitiones Analyticæ," vol. 1, Helmstadii, 1797, from Mr. S. M. Drach, F.R.A.S.

"Vorlesungen uber analytische Geometrie des Raumes," Leipzig, 1869, by Dr. Hesse.

"Vorlesungen aus der analytischen Geometrie der geraden Linie, des Punktes und des Kreises, in der Ebene," Leipzig, 1873, by the same.

"Account of the Operations of the Great Trigonometrical Survey of India. Vol. 1: the Standards of Measure and the Base Lines; also an introductory account of the early operations of the Survey during the period 1800—1830," by Col. J. T. Walker, R.E., F.R.S., Dehra Doon, 1870.

"Third and Fourth Annual Reports of the Association for the Improvement of Geometrical Teaching."

"Ueber Schaaren von quadratischen Formen," von L. Kronecker, Berlin, 1874.

"A Treatise on the Analytic Geometry of three Dimensions," by George Salmon, D.D., F.R.S., third edition, 1874.

"On the Geometrical Isomorphism of Crystals, and the derivation of all other forms from those of the Cubical System," by the Rev. W. Mitchell, M.A.

"Des Cartes, Principia Matheseos Universalis, seu Introductio ad Geometriæ methodum Renati Des Cartes, conscripta ab Er. Bartholino Casp. Fil." Editio tertia, Amstelod. 1683.

"Neutoni genesis curvarum per umbras, seu perspectivæ universalis elementa, exemplis coni sectionum et linearum tertii ordinis illustrata," Londini, 1746.

"An Essay on Magnetic Attraction, particularly as respects the Deviation of the Compass on Shipboard occasioned by the local influence of the guns, &c.," by Peter Barlow, London, 1820.

"Miscellanea Curiosa, containing a collection of some of the principal phenomena in Nature, accounted for by the greatest philosophers of this age, being the most valuable discourses read and delivered to the Royal Society," in 3 vols., 2nd edition, 1708.

"Euclidian Geometry," by F. Cuthbertson.

"Démonstration géométrique de quelques théorèmes au moyen de la considération d'une rotation infiniment petite;" and

"Construction directe du rayon de courbure de la courbe de contour apparent d'une surface qu'on projette orthogonalement sur un plan," by M. A. Mannheim.

"A new short Treatise of Algebra, with the Geometrical Construction of Equations, as far as the fourth power or dimension, together with a specimen of the Nature and Algorithm of Fluxions," by John Harris, 1714.

"Traité des progressions par addition ou des séries algébriques terminé par de nouvelles vues sur la quadrature du cercle, et précédé par un discours sur la nécessité d'un nouveau système de calcul," troisième édition, Paris, 1795.

"Ueber die Verwerthung der elliptischen Functionen für die Geometrie der Curven dritten Grades," von Axel Harnack.

"Nuove Richerche di Geometria pura sulle cubiche gobbe ed in ispecie sulla parabola gobba," par L. Cremona.

"Mémoires présentés à l'Académie des Sciences (tome xxii., No. 12) sur les surfaces trajectoires des points d'une figure de forme invariable dont le déplacement est assujetti à quatre conditions," par M. A. Mannheim.

"Ueber eine Klasse binärer Formen," by Prof. Klein.

"A Sketch in the Theory of Screws," by R. S. Ball, LL.D., 1874, (from "Hermathena," No. ii.)

"Screw Coordinates and their applications to Problems in the Dynamics of a Rigid Body," by R. S. Ball, 1874. (Trans. of Royal Irish Academy.)

"Specimen pages of a Table of the Logarithms of all Numbers up to One Million, in preparation by Edward Sang, shortened to nine figures from original calculations to fifteen places of decimals.

"Inhaltsverzeichniss der Abhandlungen der K. Akademie der Wissenschaften zu Berlin aus den Jahren 1822 bis 1872 nach den Klassen geordnet," Berlin, 1873.

"Association Française pour l'avancement des Sciences, Congrès de Lyon," 1873, Paris.

"Researches in the Dynamics of a Rigid Body by the aid of the Theory of Screws," by R. Stawell Ball, LL.D. (Phil. Trans., read June 19th, 1873).

"Sur la Fonction Exponentielle," par M. Hermite.

Cartes de Visite of Profs. Cayley, Henrici, and Wolstenholme, M. Hermite, Drs. Hesse and Ball, Messrs. G. H. Darwin, P. Frost, J. Glaisher, F.R.S., J. W. L. Glaisher, Hayward, E. J. Routh, F.R.S., and H. M. Taylor, had been received.

The Treasurer made a special statement that the sum of money presented by Lord Rayleigh had been invested in £870 Guaranteed Indian Railway Stock. The Chairman gave a sketch of what had been done in the matter from the receipt of Lord Rayleigh's letter. A cordial vote of thanks to Lord Rayleigh for his munificent donation was moved by Prof. Cayley, and seconded by the Rev. R. Harley.

At Mr. Harley's suggestion the Chairman undertook to convey by letter the thanks of the Society to his Lordship.

The Meeting then proceeded to the election of the new Council. The Scrutators (Messrs. Finlaison and Perigal), having examined the Balloting Lists, declared the following gentlemen (nominated by the Council) duly elected:

President, Prof. H. J. S. Smith, F.R.S.; Vice-Presidents, Prof. Cayley, Dr. Hirst, J. J. Sylvester, Esq., F.R.S.; Treasurer, S. Roberts, M.A.; Hon. Secs., M. Jenkins, M.A., R. Tucker, M.A.; other Members, Prof. Clifford, Messrs. T. Cotterill, J. W. L. Glaisher, Rev. R. Harley, Messrs. *Hayward, Merrifield, *W. D. Niven, Lord Rayleigh, and Mr. Spottiswoode, F.R.S. (The gentlemen marked* were not on the preceding Council list.)

Dr. John Casey was elected a Member of the Society: Mr. E. Carpmael was admitted into the Society: Mr. Harry Hart, M.A., and Mr E. J. Nanson, B.A., late and present Fellows of Trinity College, Cambridge, were proposed for election.

On the motion of Mr. Merrifield, seconded by Mr. Finlaison, it was ordered that the next Ordinary Meeting be made "Special" to consider the desirability of establishing an entrance fee, and of raising the life-composition in the case of future members.

Prof. Cayley, Vice-President, then took the chair, and Dr. Hirst, in lieu of the usual valedictory address, gave a detailed account of what he had recently done, and of what he hoped yet to do, in the subject of "Correlation in Space."

Mr. J. H. Röhrs, M.A, read a brief abstract of his Paper on "Tidal Retardation."

Prof. Wolstenholme's Paper on "the Porism of the In- and Circumscribed Triangle," was, in the author's absence, taken as read.

The following presents were received:—

Cartes de Visite of Prof. Luigi Cremona, Rev. W. H. Laverty, Mr. C. J. Monro, and Prof. A. W. Reinold.

"Crelle," 79th Band, 2es Heft.

"Bulletin des Sciences," Sept. 1874.

"Société des Sciences Physiques et naturelles de Bordeaux aux Facultés," Rue Monbazon, No. 4. "Extrait des procès-verbaux des séances," Bordeaux, 1874.

"Jahrbuch über die Fortschritte der Mathematik," vierter Band, in 3 Heften (Heft 2), Jahrgang 1872. Berlin, 1874.

"Physical Society of London, Proceedings," Part I., March 21 to June 20, 1874.

"Nautical Almanac," for 1878, from the Lords Commissioners of the Admiralty.

On Correlation in Space.

By Dr. T. A. Hirst, F.R.S.

[*Read Nov. 12th*, 1874.]

On quitting the Chair, Dr. Hirst made a communication, the object of which was to indicate how the method described in his recent paper "On the Correlation of two Planes" (see Vol. V., pp. 40—70) may be extended and applied to the corresponding, but more general and complicated question of the Correlation of two Spaces.

1. Fifteen conditions are in general necessary and sufficient to determine such a correlation. The number of solutions in each case may be most readily determined by the prior consideration of the properties of the *series* of correlations, simply infinite in number, which satisfy fourteen conditions.

2. Such a series possesses three *characteristics*, and includes a finite number of *exceptional correlations* of three distinct types.

3. The three characteristics of a series are :

μ, the number of correlations, relative to each of which two arbitrary points (one in each space) are *conjugate;* each corresponding to a plane, which passes through the other.

ρ, the number of correlations, relative to each of which two arbitrary planes (one in each space) are conjugate; each corresponding to a point which is situated in the other.

ν, the number of correlations, relative to each of which two arbitrary right lines (one in each space) are conjugate; each corresponding to a line which meets the other.

4. Of the exceptional correlations in a series,

π possess a pair of *singular points* (one in each space), whose corresponding planes are wholly indeterminate;

ω possess a pair of *singular planes* (one in each space), whose corresponding points are wholly indeterminate; and

χ possess a pair of *singular axes* (one in each space), whose corresponding lines are wholly indeterminate.

5. Between the *characteristics* and *singularities* of every series, the following three simple relations exist:

$$4\mu = \pi + 2\omega + 2\chi,$$
$$4\rho = 3\pi + \omega + 2\chi,$$
$$2\nu = \pi + \omega + 2\chi,$$

whereby μ, ρ, ν can be at once deduced, whenever, π, ω, χ have been

directly determined. The conditions being of an elementary character,* this determination can often be made with great facility; more frequently, however, it gives rise to problems of intrinsic interest, for whose solution Professor Sturm has, to some extent, prepared the way by his researches on Projectivity in Space ("Mathematische Annalen," Bd. vi., p. 513.)

6. The characteristics of a series of correlations satisfying fourteen conditions may also be thus defined:

μ indicates the class of the developable surface generated by the planes, in either space, which correspond (in the several correlations of the series) to an arbitrary point in the other space;

ρ indicates the order of the skew curve, in either space, generated by the points which correspond to an arbitrary plane in the other space; and

ν indicates the order of the scroll, in either space, generated by the lines which correspond to an arbitrary line in the other space.

7. The singular planes of each space are common to all the above *developable representatives* of points in the other space; the singular points of each space are also common to all the *skew representatives* of planes in the other space; and, lastly, the singular axes of each space are common to all the *scroll representatives* of right lines in the other space.

8. The properties of these representatives of points, planes, and lines being known, the determination of the number of correlations, in a series, which satisfy any fifteenth condition, presents little difficulty.

9. A knowledge of this number enables us, moreover, to investigate the properties of the *system* of correlations (doubly infinite in number) which satisfy thirteen given conditions.

10. Here the planes which correspond, in the several correlations of the system, to an arbitrary point in one of the two spaces, envelope a surface of known class; the points corresponding to an arbitrary plane generate another surface of known order; and the lines corresponding to an arbitrary line constitute a congruence of known order and class.

11. The exceptional correlations included in such a system are simply infinite in number. The singular planes in each space generate, in fact, a developable, which is circumscribed to every surface that represents a point in the other space; the singular points generate a skew curve, which lies on every surface that represents a plane; and

* The following are examples of such conditions:—Two given points, planes, or lines shall be conjugate (each one condition). The line corresponding to a given line shall pass through a given point, or lie in a given plane (each two conditions). To a given point (plane) shall correspond a given plane (point), (three conditions). To a given line shall correspond a given line (four conditions.)

lastly, the singular lines generate a scroll, which is common to every congruence that represents a line.

12. The special cases where the thirteen conditions are such that the surface representing an arbitrary point in either space is of the *first class*, are particularly interesting; since they furnish us with examples of the rational transformation of the two spaces, and consequently, also, of the point to point representation of surfaces on planes.

13. A fuller exposition of the method pursued, and a more complete statement of the results obtained thereby, are reserved for a future communication.

A New View of the Porism of the In- and Circum-scribed Triangle.

By J. Wolstenholme, M.A.

[*Read November 12th*, 1874.]

Starting with the system of equations

$$\frac{\tan\frac{\beta+\gamma}{2}}{\tan\alpha}=\frac{\tan\frac{\gamma+\alpha}{2}}{\tan\beta}=\frac{\tan\frac{\alpha+\beta}{2}}{\tan\gamma}=p \quad \ldots\ldots\ldots (1)$$

(which is equivalent to only a two-fold relation between α, β, γ), I propose to investigate the different forms of equations equivalent to these; and afterwards to give a geometrical interpretation, which gives a complete account of the porism of the in- and circum-scribed triangle, to a pair of coaxal conics.

The angles α, β, γ are throughout supposed unequal and less than 2π.

From (1) we get at once

$$\tan\frac{\beta-\gamma}{2}=\tan\left(\frac{\alpha+\beta}{2}-\frac{\alpha+\gamma}{2}\right)=\frac{p(\tan\gamma-\tan\beta)}{1+p^2\tan\beta\tan\gamma}$$

$$=\frac{p\sin(\gamma-\beta)}{\cos\beta\cos\gamma+p^2\sin\beta\sin\gamma},$$

or $$\cos\beta\cos\gamma+p^2\sin\beta\sin\gamma=-p\{1+\cos(\beta-\gamma)\},$$

or $$\cos\beta\cos\gamma+p\sin\beta\sin\gamma+\frac{p}{1+p}=0 \quad \ldots\ldots\ldots (2),$$

and the two like equations.

From (2) we get, since β, γ are the two roots of the equation,

$$\cos\alpha\cos\theta+p\sin\alpha\sin\theta+\frac{p}{1+p}=0,$$

$$\frac{\cos\frac{\beta+\gamma}{2}}{\cos\alpha}=\frac{\sin\frac{\beta+\gamma}{2}}{p\sin\alpha}=\frac{\cos\frac{\beta-\gamma}{2}}{-\frac{p}{1+p}};$$

whence the two systems

$$\frac{\cos\frac{\beta+\gamma}{2}}{\cos\alpha\cos\frac{\beta-\gamma}{2}}=\frac{\cos\frac{\gamma+\alpha}{2}}{\cos\beta\cos\frac{\gamma-\alpha}{2}}=\frac{\cos\frac{\alpha+\beta}{2}}{\cos\gamma\cos\frac{\alpha-\beta}{2}}=-\frac{1+p}{p}\quad\ldots\ldots(3),$$

$$\frac{\sin\frac{\beta+\gamma}{2}}{\sin\alpha\cos\frac{\beta-\gamma}{2}}=\frac{\sin\frac{\gamma+\alpha}{2}}{\sin\beta\cos\frac{\gamma-\alpha}{2}}=\frac{\sin\frac{\alpha+\beta}{2}}{\sin\gamma\cos\frac{\alpha-\beta}{2}}=-(1+p)\quad\ldots(4).$$

Next consider the system

$$\cot\frac{\beta}{2}\cot\frac{\gamma}{2}+\cot\frac{\gamma}{2}\cot\frac{\alpha}{2}+\cot\frac{\alpha}{2}\cot\frac{\beta}{2}$$

$$=\tan\frac{\beta}{2}\tan\frac{\gamma}{2}+\tan\frac{\gamma}{2}\tan\frac{\alpha}{2}+\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=q,$$

which give the equation in β, γ,

$$\left(\tan\frac{\beta}{2}+\tan\frac{\gamma}{2}\right)\left(\cot\frac{\beta}{2}+\cot\frac{\gamma}{2}\right)$$

$$=\left(q-\tan\frac{\beta}{2}\tan\frac{\gamma}{2}\right)\left(q-\cot\frac{\beta}{2}\cot\frac{\gamma}{2}\right),$$

or $$4\sin^2\frac{\beta+\gamma}{2}=\left\{(q-1)\cos\frac{\beta-\gamma}{2}+(q+1)\cos\frac{\beta+\gamma}{2}\right\}$$

$$\times\left\{(q-1)\cos\frac{\beta-\gamma}{2}-(q+1)\cos\frac{\beta+\gamma}{2}\right\},$$

or $$4\{1-\cos(\beta+\gamma)\}=(q-1)^2\{1+\cos(\beta-\gamma)\}$$

$$-(q+1)^2\{1+\cos(\beta+\gamma)\},$$

or $$2(q-1)\cos\beta\cos\gamma+(1-q^2)\sin\beta\sin\gamma+2(1+q)=0,$$

which is equivalent to (2) if

$$p=-\frac{1+q}{2},\quad\text{or}\quad q=-(2p+1).$$

Hence, the system

$$\cot\frac{\beta}{2}\cot\frac{\gamma}{2}+\cot\frac{\gamma}{2}\cot\frac{\alpha}{2}+\cot\frac{\alpha}{2}\cot\frac{\beta}{2}$$

$$=\tan\frac{\beta}{2}\tan\frac{\gamma}{2}+\tan\frac{\gamma}{2}\tan\frac{\alpha}{2}+\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=-(2p+1)\quad\ldots(5).$$

One of the system (1) gives us

$$\tan\frac{\gamma+\alpha}{2}\tan\gamma=\tan\frac{\alpha+\beta}{2}\tan\beta;$$

whence $$\frac{\cos\frac{\gamma-\alpha}{2}}{\cos\frac{3\gamma+\alpha}{2}} = \frac{\cos\frac{\beta-\alpha}{2}}{\cos\frac{3\beta+\alpha}{2}},$$

or $$\cos\frac{3\beta+\gamma}{2} + \cos\left(\frac{3\beta-\gamma}{2}+\alpha\right) = \cos\frac{3\gamma+\beta}{2} + \cos\left(\frac{3\gamma-\beta}{2}+\alpha\right),$$

or $$2\sin\frac{\beta-\gamma}{2}\sin(\beta+\gamma) = \sin(\gamma-\beta)\sin\left(\frac{\beta+\gamma}{2}+\alpha\right),$$

or $$\sin(\beta+\gamma) + 2\cos\frac{\beta-\gamma}{2}\sin\left(\frac{\beta+\gamma}{2}+\alpha\right) = 0,$$

or $$\sin(\beta+\gamma) + \sin(\gamma+\alpha) + \sin(\alpha+\beta) = 0.$$

Also, if $$x = \tan\frac{\alpha}{2}, \quad y = \tan\frac{\beta}{2}, \quad z = \tan\frac{\gamma}{2},$$

$$\cos\frac{\beta-\gamma}{2}\cos\frac{\gamma-\alpha}{2}\cos\frac{\alpha-\beta}{2} = \frac{(1+yz)(1+zx)(1+xy)}{(1+x^2)(1+y^2)(1+z^2)};$$

and since $$yz+zx+xy = \frac{1}{yz}+\frac{1}{zx}+\frac{1}{xy} = q,$$

the numerator $= (1+q)(1+x^2y^2z^2)$,

and the denominator

$$= 1+(q^2x^2y^2z^2-2q)+(q^2-2qx^2y^2z^2)+x^2y^2z^2 = (1-q)^2(1+x^2y^2z^2),$$

or $$\cos\frac{\beta-\gamma}{2}\cos\frac{\gamma-\alpha}{2}\cos\frac{\alpha-\beta}{2} = \frac{1+q}{(1-q)^2} = -\frac{p}{2(1+p)^2}.$$

Hence the system $$\left.\begin{aligned}\sin(\beta+\gamma)+\sin(\gamma+\alpha)+\sin(\alpha+\beta) &= 0\\ 2\cos\frac{\beta-\gamma}{2}\cos\frac{\gamma-\alpha}{2}\cos\frac{\alpha-\beta}{2} &= \frac{-p}{(1+p)^2}\end{aligned}\right\} \ldots\ldots (6).$$

Since $\sin(\beta+\gamma)+\sin(\gamma+\alpha)+\sin(\alpha+\beta) = 0$, we have

$$\sin(\alpha+\beta+\gamma)(\cos\alpha+\cos\beta+\cos\gamma) - \cos(\alpha+\beta+\gamma)(\sin\alpha+\sin\beta+\sin\gamma) = 0,$$

and we may write $$\cos\alpha+\cos\beta+\cos\gamma = m\cos(\alpha+\beta+\gamma),$$
$$\sin\alpha+\sin\beta+\sin\gamma = m\sin(\alpha+\beta+\gamma);$$

and multiplying by $\cos\alpha$, $\sin\alpha$, and adding, we get

$$1+\cos(\alpha-\beta)+\cos(\alpha-\gamma) = m\cos(\beta+\gamma);$$

whence $$1+\cos(\beta-\gamma)+\cos(\gamma-\alpha)+\cos(\alpha-\beta)$$
$$= \cos(\beta-\gamma)+m\cos(\beta+\gamma)$$
$$= 4\cos\frac{\beta-\gamma}{2}\cos\frac{\gamma-\alpha}{2}\cos\frac{\alpha-\beta}{2} = -\frac{2p}{(1+p)^2},$$

which will coincide with (2) if

$$\frac{1-m}{1+m} = p, \quad \text{or} \quad m = \frac{1-p}{1+p}.$$

Hence
$$\left.\begin{aligned} \cos\alpha+\cos\beta+\cos\gamma &= \frac{1-p}{1+p}\cos(\alpha+\beta+\gamma) \\ \sin\alpha+\sin\beta+\sin\gamma &= \frac{1-p}{1+p}\sin(\alpha+\beta+\gamma) \end{aligned}\right\} \quad \ldots\ldots\ldots (7).$$

This leads also to $\cos(\beta+\gamma)+\cos(\gamma+\alpha)+\cos(\alpha+\beta)$

$$= \cos(\alpha+\beta+\gamma)(\cos\alpha+\cos\beta+\cos\gamma)$$
$$+\sin(\alpha+\beta+\gamma)(\sin\alpha+\sin\beta+\sin\gamma) = \frac{1-p}{1+p};$$

and therefore, by (2), since

$$\cos\beta\cos\gamma+\cos\gamma\cos\alpha+\cos\alpha\cos\beta+p(\sin\beta\sin\gamma+\ldots) = -\frac{3p}{1+p},$$

we have
$$\left.\begin{aligned} \cos\beta\cos\gamma+\cos\gamma\cos\alpha+\cos\alpha\cos\beta &= -\frac{p(p+2)}{(1+p)^2} \\ \sin\beta\sin\gamma+\sin\gamma\sin\alpha+\sin\alpha\sin\beta &= -\frac{2p+1}{(1+p)^2} \end{aligned}\right\} \quad \ldots\ldots (8).$$

Again, from (2), multiplying by $\cos\alpha$, $\cos\beta$, $\cos\gamma$, and adding, we have

$$3\cos\alpha\cos\beta\cos\gamma+p\{\cos\alpha\cos\beta\cos\gamma-\cos(\alpha+\beta+\gamma)\}$$
$$= -\frac{p}{1+p}(\cos\alpha+\cos\beta+\cos\gamma),$$

or
$$(3+p)\cos\alpha\cos\beta\cos\gamma = \left(p-\frac{p(1-p)}{(1+p)^2}\right)\cos(\alpha+\beta+\gamma)$$
$$= \frac{p^2(3+p)}{(1+p)^2}\cos(\alpha+\beta+\gamma),$$

or
$$\cos\alpha\cos\beta\cos\gamma = \frac{p^2}{(1+p)^2}\cos(\alpha+\beta+\gamma)$$

and similarly
$$\sin\alpha\sin\beta\sin\gamma = -\frac{1}{(1+p)^2}\sin(\alpha+\beta+\gamma) \quad \ldots\ldots (9).$$

I may observe that, since the original equations (1) are unaltered, if we write $\frac{\pi}{2}-\alpha$, $\frac{\pi}{2}-\beta$, $\frac{\pi}{2}-\gamma$ for α, β, γ, and $\frac{1}{p}$ for p, we may deduce several of the above equations by these substitutions.

Next consider the equation in θ,

$$p\frac{\cos\phi}{\cos\theta}+\frac{\sin\phi}{\sin\theta} = -(1+p),$$

a biquadratic, of which one root is $\pi+\phi$, and for the other three we

have $$p\cos\frac{\phi+\theta}{2}\sin\theta+\sin\frac{\phi+\theta}{2}\cos\theta=0,$$

or $$p\left(1-z\tan\frac{\phi}{2}\right)2z+\left(z+\tan\frac{\phi}{2}\right)(1-z^2)=0,$$

writing $z=\tan\frac{\theta}{2}$,

or $$z^3+z^2(2p+1)\tan\frac{\phi}{2}-(2p+1)z-\tan\frac{\phi}{2}=0;$$

whence we see, by (5), that the roots may be taken to be α, β, γ; and if α, β, γ, δ be the four roots of the original equation in θ, it is readily proved that $\alpha+\beta+\gamma+\delta=\pi$ (or an odd multiple of π), so that $\alpha+\beta+\gamma+\phi=0$ (or $2r\pi$); whence we see also that

$$\left.\begin{aligned}\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}+\tan\frac{\gamma}{2}&=-(2p+1)\tan\frac{\phi}{2}\\&=(2p+1)\tan\frac{\alpha+\beta+\gamma}{2}\\ \text{and}\quad\cot\frac{\alpha}{2}+\cot\frac{\beta}{2}+\cot\frac{\gamma}{2}&=-(2p+1)\cot\frac{\phi}{2}\\&=(2p+1)\cot\frac{\alpha+\beta+\gamma}{2}\end{aligned}\right\}\quad\ldots\ldots(10).$$

A similar relation between the tangents will follow at once from (7), since

$$\left.\begin{aligned}\tan\alpha+\tan\beta+\tan\gamma&=\frac{\sin\alpha\cos\beta\cos\gamma+\ldots}{\cos\alpha\cos\beta\cos\gamma}\\&=\frac{\sin(\alpha+\beta+\gamma)+\sin\alpha\sin\beta\sin\gamma}{\cos\alpha\cos\beta\cos\gamma}\\&=\tan(\alpha+\beta+\gamma)\,\frac{1-\dfrac{1}{(1+p)^2}}{\dfrac{p^2}{(1+p)^2}}=\frac{p+2}{p}\tan(\alpha+\beta+\gamma)\\ \text{and}\quad\cot\alpha+\cot\beta+\cot\gamma&=(2p+1)\cot(\alpha+\beta+\gamma)\end{aligned}\right\}\quad\ldots\ldots(11).$$

These, however, would be better deduced from the equation in $\tan\theta$, deduced from $\tan\frac{\phi+\theta}{2}=-p\tan\theta$, which leads to ($z\equiv\tan\theta$, $c\equiv\tan\phi$),

$$p^2z^3+p(p+2)cz^2-(2p+1)z-c=0,$$

giving the two preceding, and also

$$\left.\begin{aligned}\tan\beta\tan\gamma+\tan\gamma\tan\alpha+\tan\alpha\tan\beta&=-\frac{2p+1}{p^2}\\ \cot\beta\cot\gamma+\cot\gamma\cot\alpha+\cot\alpha\cot\beta&=-p(p+2)\end{aligned}\right\}\quad\ldots\ldots(12).$$

It is probable that relations of the same form would be found for the

tangents and cotangents of 2α, 2β, 2γ, &c., and the number of different forms in which the system may be written seems to be unlimited.

Now suppose along the normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, at the point $a\cos z$, $b\sin z$, we measure inwards a length PQ equal to k times the corresponding conjugate semi-diameter CD; the coordinates of Q will be $(a-kb)\cos z$, $(b-ka)\sin z$, and the equation determining the normals which can be drawn from this point will be

$$a(a-kb)\frac{\cos z}{\cos\theta}-b(b-ka)\frac{\sin z}{\sin\theta}=a^2-b^2,$$

and the roots of this equation, other than z, will be α, β, γ, if $p=-\frac{a(a-kb)}{b(b-ka)}$, the z of this equation differing by π from the ϕ of the former, so that $\alpha+\beta+\gamma=\pi-z$. The straight lines joining the points α, β, γ will all touch the ellipse whose semi-axes are

$$a'=\frac{a^2}{a^2-b^2}(a-kb),\quad b'=\frac{b^2}{b^2-a^2}(b-ka),$$

the coordinates of a point of contact being $a'\cos(\pi+\alpha)$, $b'\sin(\pi+\alpha)$; and the tangents at β, γ will intersect in the point $\mathrm{A}\cos(\pi+\alpha)$, $\mathrm{B}\sin(\pi+\alpha)$, where $\mathrm{A}=\frac{a^2}{a'}$, $\mathrm{B}=\frac{b^2}{b'}$. I have thought it better to take such values for a', b' that the equation connecting them may be $\frac{a'}{a}+\frac{b'}{b}=1$. This may give negative values for the lengths of the axes, but the coordinates of the angular points of the several triangles will be in all cases correct. Thus the tangents at β, γ intersect in the

point $$\frac{a\cos\frac{\beta+\gamma}{2}}{\cos\frac{\beta-\gamma}{2}},\quad \frac{b\sin\frac{\beta+\gamma}{2}}{\cos\frac{\beta-\gamma}{2}},$$

which, by (3), (4), is the point

$$-\frac{(1+p)}{p}\cos\alpha,\quad -(1+p)\sin\alpha,$$

or $$\frac{a^2-b^2}{(a-kb)}\cos(\pi+\alpha),\quad \frac{b^2-a^2}{b-ka}\sin(\pi+\alpha),$$

which is the point stated above. All cases of the in- and circumscribed triangles to two coaxal conics are particular cases of this. The following theorems are only geometrical interpretations of equations already investigated.

If ABC be a triangle inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, so that

the normals at A, B, C meet in a point Q, and from Q the fourth normal QP be drawn, QP will bear to the semi-diameter CD conjugate to CP, the ratio $k : 1$, k being found from either of the equations

$$X = -(a-kb)\cos(\alpha+\beta+\gamma), \quad Y = (b-ka)\sin(\alpha+\beta+\gamma),$$

where (X, Y) are the coordinates of Q, and α, β, γ the excentric angles of ABC.

The sides of the triangle ABC will touch the ellipse $\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1$ in points whose excentric angles are $\pi+\alpha$, $\pi+\beta$, $\pi+\gamma$, and

$$a' = \frac{a^2}{a^2-b^2}(a-kb), \quad b' = \frac{b^2}{b^2-a^2}(b-ka).$$

The tangents at A, B, C will form a triangle whose angular points lie on the ellipse $\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$, and whose excentric angles are $\pi+\alpha$, $\pi+\beta$, $\pi+\gamma$; A being $= \frac{a^2}{a'}$, $B = \frac{b^2}{b'}$.

An infinite number of such triangles ABC can be inscribed to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and circumscribed to the ellipse $\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1$, the excentric angles of the points A, B, C satisfying all the relations investigated, any two of which involve all the others. Hence the ratio k remains the same for all these triangles; and if A′, B′, C′ be the points of contact, the ratio of the areas of the triangles A′B′C′, ABC is always the same, being $a'b' : ab$ or $\frac{ab(a-kb)(b-ka)}{(a^2-b^2)^2} : 1$.

The four points related to each triangle—(1) the centroid, (2) the centre of perpendiculars, (3) the centre of the circumscribed circle, (4) the point of concourse of the normals—all lie on fixed ellipses coaxal with the original, and the excentric angle is always the excess of the sum of the excentric angles of A, B, C above π; while the semi-axes are

(1) $\frac{a(a^2+b^2-2kab)}{3(a^2-b^2)}$, $\frac{b(a^2+b^2-2kab)}{3(a^2-b^2)}$;

(2) $\frac{(a^2+b^2)(a^2+b^2-2kab)+(a^2-b^2)^2}{2a(a^2-b^2)}$,

$\frac{(a^2+b^2)(a^2+b^2-2kab)+(a^2-b^2)^2}{2b(a^2-b^2)}$;

(3) $\frac{b(b-ka)}{2a}$, $\frac{a(kb-a)}{2b}$; (4) $a-kb$, $ka-b$.

Of course also the corresponding points for the triangle A′B′C′, and for the triangle formed by the tangents at A, B, C, will trace out cor-

responding ellipses which may be found by writing a', b', k' for a, b, k, where

$$a' = \frac{a^2}{a^2-b^2}(a-kb), \quad b' = \frac{b^2}{b^2-a^2}(b-ka), \quad \frac{a'(a'-k'b')}{b'(b'-k'a')} = \frac{a(a-kb)}{b(b-ka)},$$

whence $$k+k' = \frac{a^2+b^2}{ab};$$

or, by writing A, B, K for a, b, k, where $A = \frac{a^2}{a'}$, $B = \frac{b^2}{b'}$,

$$\frac{A(A-KB)}{B(B-KA)} = \frac{a(a-kb)}{b(b-ka)},$$

whence $$Kk = 1 - \frac{ab(1-k^2)^2}{(a-kb)(b-ka)}.$$

If the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ be given, the envelope of the coaxal ellipses to which the triangles are circumscribed is $\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1$; and that of the ellipses, on which lie the corners of the triangles formed by the tangents at A, B, C, is $\frac{a^2}{x^2} + \frac{b^2}{y^2} = 1$. The envelope of the locus of the point of concourse of the normals at ABC is of course the evolute of the given ellipse, since at a point on the evolute two values of k will become equal.

If the tangents at ABC form a triangle $A_1B_1C_1$, and the tangents at $A_1B_1C_1$ to their locus a triangle $A_2B_2C_2$, and so on, we shall have for the locus of $A_nB_nC_n$ an ellipse whose axes are $a\lambda^n$, $b\mu^n$, where

$$\lambda = \frac{A}{a} = \frac{a^2-b^2}{a(a-kb)}, \quad \mu = \frac{b^2-a^2}{b(b-ka)}, \quad \text{so that} \quad \frac{1}{\lambda} + \frac{1}{\mu} = 1,$$

and if k_n be the value of the quantity corresponding to k in this ellipse

$$\frac{a\lambda^n(a\lambda^n - k_n b\mu^n)}{b\mu^n(b\mu^n - k_n a\lambda^n)} = \frac{a(a-kb)}{b(b-ka)} = -\frac{\mu}{\lambda},$$

or $$k_n = \frac{a^2\lambda^{2n+1} + b^2\mu^{2n+1}}{(\lambda\mu)^n(\lambda+\mu)ab}.$$

Sufficient equations have been found to determine the loci and envelopes of any inscribed and circumscribed triangles, and of the points connected with them. Perhaps the centre of the Nine Points' Circle might have been added, but as this is the centre of the line joining the points (1) and (2), its locus will manifestly be of the same kind.

As it may interest some readers to see equivalent algebraical equations similarly treated, I append, at Prof. Cayley's suggestion, the greater number of the results above obtained, now deduced algebraically from a single system of three equations, equivalent, of course, to

two only. If we write x, y, z for $\tan\frac{\alpha}{2}$, $\tan\frac{\beta}{2}$, $\tan\frac{\gamma}{2}$ throughout, we shall get a series of algebraical systems of equations, each equivalent to two relations only. The original system (1) becomes

$$\left.\begin{aligned}(1-x^2)(y+z) &= 2p(x-xyz)\\ (1-y^2)(z+x) &= 2p(y-xyz)\\ (1-z^2)(x+y) &= 2p(z-xyz)\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (\alpha);$$

and multiplying these by $y-z$, $z-x$, $x-y$, and adding, we have an identity, proving that the system is equivalent to a two-fold relation only. Subtracting, we get from any pair

$$yz+zx+xy = -(2p+1) \equiv -q;$$

and multiplying the second by z, and the third by y, and subtracting, we get

$$\frac{1}{yz}+\frac{1}{zx}+\frac{1}{xy} = -(2p+1) \quad \ldots\ldots\ldots\ldots\ldots (\beta).$$

These two, giving the relations between x, y, z in the simplest form, are equivalent to (5).

If we eliminate x between these, we shall get a relation which must be equivalent to the first of (2), and we thus have the system

$$\left.\begin{aligned}(y+z)^2 &= (q+yz)(1+qyz)\\ (z+x)^2 &= (q+zx)(1+qzx)\\ (x+y)^2 &= (q+xy)(1+qxy)\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (\gamma).$$

From the last two of these (showing that y, z are the two roots of a certain quadratic), we have

$$\frac{1}{1-qx^2} = \frac{y+z}{x(q^2-1)} = \frac{yz}{x^2-q},$$

and therefore

$$= \frac{1+yz}{(1+x^2)(1-q)} = \frac{1-yz}{(1-x^2)(1+q)},$$

or

$$\frac{(1-yz)(1+x^2)}{(1+yz)(1-x^2)} = \frac{1+q}{1-q} = -\frac{p+1}{p},$$

and the system

$$\frac{(1-yz)(1+x^2)}{(1+yz)(1-x^2)} = \frac{(1-zx)(1+y^2)}{(1+zx)(1-y^2)} = \frac{(1-xy)(1+z^2)}{(1+xy)(1-z^2)} = -\frac{1+p}{p} \quad \ldots\ldots\ldots\ldots\ldots (\delta),$$

equivalent to (3).

Combining this with (α), we get

$$\frac{(y+z)(1+x^2)}{2x(1+yz)} = \frac{(z+x)(1+y^2)}{2y(1+zx)} = \frac{(x+y)(1+z^2)}{2z(1+xy)} = -(1+p) \ldots (\epsilon),$$

equivalent to (4).

Of course any of these systems of three equations are equivalent to

two only; and, any two being assumed, the third is readily deducible from them.

The system (6) is equivalent to

$$\left.\begin{aligned}(y+z)(1-yz)(1+x^2)+(z+x)(1-zx)(1+y^2)\\ +(x+y)(1-xy)(1+z^2)=0\\ (1+yz)(1+zx)(1+xy)=-\frac{p}{2(1+p)^2}(1+x^2)(1+y^2)(1+z^2)\end{aligned}\right\}\ \ldots(\zeta),$$

of which the first is identically

$$x+y+z=xyz(yz+zx+xy),$$

and the second was proved algebraically before.

If we denote $x+y+z$, $yz+zx+xy$, and xyz by a, b, c, the given relations are equivalent to $b=\frac{a}{c}=-(2p+1)$. Expressing the fraction

$$\frac{(1-x^2)(1+y^2)(1+z^2)+(1-y^2)(1+z^2)(1+x^2)+(1-z^2)(1+x^2)(1+y^2)}{(1-x^2)(1-y^2)(1-z^2)-4yz(1-x^2)-4zx(1-y^2)-4xy(1-z^2)}$$

in terms of a, b, c, and then substituting for a its equivalent bc, this becomes $\dfrac{(3-2b-b^2)(1-c^2)}{(1-b)^2(1-c^2)}\left(=\dfrac{3+b}{1-b}=\dfrac{1-p}{1+p}\right)$.

Similarly

$$\frac{2x(1+y^2)(1+z^2)+2y(1+z^2)(1+x^2)+2z(1+x^2)(1+y^2)}{-8xyz+2x(1-y^2)(1-z^2)+2y(1-z^2)(1-x^2)+2z(1-x^2)(1-y^2)}$$

$$=\frac{2c(b^2+2b-3)}{-2c(b-1)^2}=\frac{1-p}{1+p},$$

which are the equivalents of

$$\frac{\cos\alpha+\cos\beta+\cos\gamma}{\cos(\alpha+\beta+\gamma)}=\frac{\sin\alpha+\sin\beta+\sin\gamma}{\sin(\alpha+\beta+\gamma)}=\frac{1-p}{1+p}.$$

In the same manner,

$$\cos\beta\cos\gamma+\cos\gamma\cos\alpha+\cos\alpha\cos\beta$$

$$=\frac{(1+x^2)(1-y^2)(1-z^2)+\text{two like terms}}{(1+x^2)(1+y^2)(1+z^2)}$$

$$=\frac{3-(x^2+y^2+z^2)-(y^2z^2+z^2x^2+x^2y^2)+3x^2y^2z^2}{1+(\ldots)+(\ldots)+x^2y^2z^2}.$$

Now

$$x^2+y^2+z^2=a^2-2b=b^2c^2-2b,$$

$$y^2z^2+z^2x^2+x^2y^2=b^2-2ac=b^2-2bc^2,$$

so that the fraction

$$=\frac{(1+c^2)(3+2b-b^2)}{(1+c^2)(1-2b+b^2)}=\frac{(1+b)(3-b)}{(1-b)^2}=-\frac{p(p+2)}{(1+p)^2};$$

$$\text{and} \quad \sin\beta\sin\gamma + \ldots = \frac{4yz(1+x^2)+\ldots}{(1+x^2)(1+y^2)(1+z^2)} = \frac{4(b+ac)}{(1+c^2)(1-b)^2}$$

$$= \frac{4b}{(1-b)^2} = -\frac{(2p+1)}{(1+p)^2},$$

$$\frac{\cos\alpha\cos\beta\cos\gamma}{\cos(\alpha+\beta+\gamma)} = \frac{(1-x^2)(1-y^2)(1-z^2)}{(1-x^2)(1-y^2)(1-z^2)-4yz(1-x^2)-\ldots}$$

$$= \frac{1-b^2c^2+2b+b^2-2bc^2-c^2}{(1-b)^2(1-c^2)} = \left(\frac{1+b}{1-b}\right)^2 = \left(\frac{p}{1+p}\right)^2,$$

$$\text{and} \quad \frac{\sin\alpha\sin\beta\sin\gamma}{\sin(\alpha+\beta+\gamma)} = \frac{8xyz}{-8xyz+2x(1-y^2)(1-z^2)+\ldots}$$

$$= \frac{8c}{-8c+2a-2(ab-3c)+2bc} = \frac{8c}{-2c+4bc-2b^2c}$$

$$= \frac{-4}{(1+b)^2} = -\frac{1}{(1+p)^2}.$$

Many more such equivalent systems might be obtained, but I think the above are the most remarkable. The great simplification of some of the fractions, by the use of the single relation

$$x+y+z = xyz(yz+zx+xy),$$

is certainly deserving of attention.

December 10th, 1874.

Prof. H. J. S. SMITH, F.R.S., President, in the Chair.

SPECIAL MEETING.

The Chairman having stated the purposes for which the Meeting had been made a "Special" one, in accordance with the Resolution carried at the November Meeting, called upon Mr. Merrifield to move the two following Resolutions:—

"That in future there shall be an entrance-fee of one guinea; and

"That the life-composition shall be raised from £10, its present amount, to fifteen guineas."

It was resolved that the Resolutions should be taken separately. After some discussion, in which Mr. Harley, Dr. Hirst, and others took part, the first Resolution was carried by a large majority.

A protracted discussion followed upon the second Resolution; and at last an amendment, proposed by Mr. Harley, and seconded by Mr.

J. W. L. Glaisher, was carried. This amendment recommended that the life-composition be changed from £10 to ten guineas (on account of the annual subscription being now one guinea).

On the motion of Mr. Harley, seconded by Dr. Hirst, Rule 36* was ordered to be abolished.

The proposal to substitute "session" for "year," in Rule 19, was, at the suggestion of Mr. Stirling, ordered to be deferred, that the bearing of the proposed alteration upon some of the other rules might be considered.

The Meeting then became a "General" one, and Mr. W. D. Niven was admitted into the Society. Messrs. Hart and Nanson were elected Members of the Society, and the following gentlemen were nominated for Membership:—John Wesley Russell, B.A., Fellow of Merton College, Oxford; Charles M. Leudesdorf, B.A., Fellow of Pembroke College, Oxford; Edwin Bailey Elliott, B.A., Fellow of Queen's College, Oxford; H. M. Jeffery, M.A., of Cheltenham; Charles Smith, M.A., Fellow of Sidney Sussex College, Cambridge; and Benjamin Williamson, M.A., Fellow and Tutor of Trinity College, Dublin.

The Auditor (Mr. Stirling) stated that he had examined the Treasurer's accounts, and found them perfectly correct.

The Chairman, on the recommendation of the Council, nominated Drs. Klein, Kronecker, and Zeuthen, for the honour of Foreign Membership.

Prof. Cayley then read his paper on "the Potentials of Polygons and Polyhedra."

Mr. Tucker (in the absence of J. J. Sylvester, Esq., F.R.S.) gave a sketch of the contents of two letters from M. Mannheim, on "Three and Seven Bar Motion."

The following presents were received:—

"Memoir on the Transformation of Elliptic Functions," by Prof. Cayley: from the Author (from the Phil. Trans., read Jan. 8, 1874).

Carte de Visite of Mr. J. Walmsley.

On the Potentials of Polygons and Polyhedra. By Prof. Cayley.

[*Read December 10th*, 1874.]

The problem of the attraction of polyhedra is treated of by Mehler, "Crelle," t. 66, pp. 375—381 (1866); but the results here obtained are exhibited under forms which are very different from his, and which give rise to further developments of the theory.

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* Rule 36.—At no two successive Meetings shall the papers be entirely on Applied Mathematics.

General Formulæ for the Potentials of a Cone and a Shell.

1. The law of attraction is taken to be according to the inverse square of the distance; and I commence with the general case of a cone standing upon any portion of a surface Σ as its base, and attracting a point at its vertex, the cone being considered as a mass of density unity.

2. Considering, in the first instance, an element of mass, the position of which is determined by its distance r from the vertex (or origin) and by two angular coordinates defining the position of the radius vector r, then the element is $= r^2 dr\, d\omega$ (where $d\omega$ is the element of solid angle, or surface of the unit-sphere), and the corresponding element of potential is $\frac{1}{r} r^2 dr\, d\omega, = r\, dr\, d\omega$; whence

$$V = \int r\, dr\, d\omega,$$

which, integrating from $r = 0$ to $r =$ its value at the surface, is

$$= \tfrac{1}{2} \int r^2 d\omega,$$

where r now denotes the radius vector at a point of the surface, being, therefore, a given function of the two angular coordinates: and the remaining (double) integration is to be extended to all values of the angular coordinates belonging to a position of r within the conical surface which is the other boundary of the attracting mass, or say over the spherical aperture of the cone.

3. If the value of the radius vector at the surface is taken to be mr (m a constant), then we have obviously

$$V = \tfrac{1}{2} m^2 \int r^2 d\omega;$$

and hence also, writing $m + dm$ instead of m, we obtain, for the potential of the portion of the shell lying between the similar and similarly situated surfaces Σ, Σ', belonging to the parameters m and $m + dm$ respectively, the value

$$V = m\, dm \int r^2 d\omega;$$

this is $= 2 \frac{dm}{m}$ into the potential of the cone; and we thus see that it is the same problem to determine the potential of the cone, and that of the subtended portion of the indefinitely thin shell included between the two surfaces.

4. The same result may be arrived at as follows: the element of solid angle $d\omega$ determines on the surface an element of surface $d\Sigma$, and if dv be the corresponding normal thickness of the shell, then

the element of mass is $= d\nu\, d\Sigma$, and the element of potential is $= \frac{1}{mr} d\nu\, d\Sigma$ (mr being, as before, the radius vector at the surface). Take α the complement of the inclination of the radius vector to the tangent plane—that is, α the inclination of the radius vector to the normal, or, what is the same thing, to the perpendicular from the origin on the tangent plane (whence, also, if mp be the length of this perpendicular, then $p = r\cos\alpha$). The shell-thickness in the direction of the radius vector is $= r\,dm$, or we have $d\nu = r\,dm\cos\alpha$; the element of potential is therefore $= \frac{dm}{m}\cos\alpha\, d\Sigma$. But $d\omega$ being the spherical aperture of the cone subtending the element $d\Sigma$, the perpendicular section at the distance mr is $= m^2 r^2 d\omega$; we have therefore $d\Sigma = \frac{1}{\cos\alpha} m^2 r^2 d\omega$; and hence the element of potential is $= m\,dm\,.\,r^2 d\omega$, or the potential of the subtended portion of the shell is as before, $= m\,dm \int r^2 d\omega$.

5. It may be added that, integrating between the values m, n ($m > n$), we obtain $\frac{1}{2}(m^2 - n^2) \int r^2 d\omega$ for the potential of the shell-portion included between the surfaces mr, nr; and if $n = 0$, then, as before, the potential of the cone is $= \frac{1}{2} m^2 \int r^2 d\omega$.

Cone on a plane Base, and plane Figure.

6. Suppose that the surface Σ is a plane, the surface Σ' is, of course, a parallel plane. Taking here mp for the perpendicular distance of the plane Σ from the origin; then, if δ be the infinitesimal distance of the two planes from each other, we have $\delta = p\,dm$, that is $dm = \frac{\delta}{p}$; the potential of the cone is, as before, $= \frac{1}{2} m^2 \int r^2 d\omega$, and that of the plane figure, thickness δ, is $= \frac{m\delta}{p} \int r^2 d\omega$.

7. Taking, for greater convenience, $m = 1$, we have

$$\text{Potential of cone} = \tfrac{1}{2} \int r^2 d\omega,$$

$$\text{Do. of plane figure} = \frac{\delta}{p} \int r^2 d\omega,$$

where p is now the perpendicular distance of the plane from the vertex; or if, as regards the plane figure, the infinitesimal thickness δ is taken as unity, then

$$\text{Potential of plane figure} = \frac{1}{p} \int r^2 d\omega;$$

and in each case r is the value of the radius vector corresponding to a point of the plane figure which is the base of the cone, and the integration extends over the spherical aperture of the cone.

8. If the position of the radius vector is determined by the usual angular coordinates, θ its inclination to the axis of z, and ϕ its azimuth from the plane of zx—viz., if we have

$$x = r\sin\theta\cos\phi,$$
$$y = r\sin\theta\sin\phi,$$
$$z = r\cos\theta;$$

then, as is well known, $d\omega = \sin\theta\, d\theta\, d\phi$, and the integral $\int r^2 d\omega$ is $= \int r^2 \sin\theta\, d\theta\, d\phi$.

Taking the inclination of p to the axes to be α, β, γ respectively, the equation of the plane which is the base of the cone is

$$x\cos\alpha + y\cos\beta + z\cos\gamma = p;$$

viz., we have

$$r[(\cos\alpha\cos\phi + \cos\beta\sin\phi)\sin\theta + \cos\gamma\cos\theta] = p;$$

that is, $$r = \frac{p}{(\cos\alpha\cos\phi + \cos\beta\sin\phi)\sin\theta + \cos\gamma\cos\theta},$$

and the integral $\int r^2 d\omega$ is therefore

$$= p^2 \int \frac{\sin\theta\, d\theta\, d\phi}{[(\cos\alpha\cos\phi + \cos\beta\sin\phi)\sin\theta + \cos\gamma\cos\theta]^2};$$

and, in particular, if p coincide with the axis of z, so that the equation of the plane is $z = p$, then the integral is

$$= p^2 \int \frac{\sin\theta\, d\theta\, d\phi}{\cos^2\theta}.$$

9. The integration in regard to θ can be at once performed; viz., in the latter case we have $\int \frac{\sin\theta\, d\theta}{\cos^2\theta} = \sec\theta$; and in the former case, writing, as we may do,

$$(\cos\alpha\cos\phi + \cos\beta\sin\phi)\sin\theta + \cos\gamma\cos\theta = \mathrm{M}\cos(\theta - \mathrm{N}),$$

then

$$\int \frac{\sin\theta\, d\theta}{[(\cos\alpha\cos\phi + \cos\beta\sin\phi)\sin\theta + \cos\gamma\cos\theta]^2} = \frac{1}{\mathrm{M}^2}\int \frac{\sin(\theta - \mathrm{N} + \mathrm{N})\, d\theta}{\cos^2(\theta - \mathrm{N})}$$

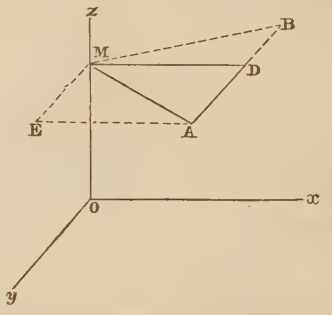
$$= \frac{1}{\mathrm{M}^2}\left[\cos\mathrm{N}\int \frac{\sin(\theta-\mathrm{N})\, d\theta}{\cos^2(\theta-\mathrm{N})} + \sin\mathrm{N}\int \frac{d\theta}{\cos(\theta-\mathrm{N})}\right]$$

$$= \frac{1}{\mathrm{M}^2}\left(\cos\mathrm{N}\sec(\theta-\mathrm{N}) + \sin\mathrm{N}\log\tan\left\{\tfrac{1}{4}\pi + \tfrac{1}{2}(\theta-\mathrm{N})\right\}\right).$$

Case of a Polyhedron or a Polygon.

10. Consider now the pyramid, vertex the origin O, standing on a polygonal base. Letting fall from the vertex a perpendicular OM on the base of the pyramid, and drawing planes through OM and the several vertices of the polygon, we thus divide the pyramid into triangular pyramids; viz., AB being any side of the polygon, a component pyramid (or tetrahedron) will be OMAB, vertex O and base MAB, where MO is a perpendicular at M to the triangular base MAB. And drawing through MO a plane at right angles to AB, meeting it in D (viz., MD is the perpendicular from M on the base AB of the triangle), we divide the triangular pyramid into two pyramids OMAD, OMBD, each having for its base a right-angled triangle; viz., the vertex is O, and the base is the triangle ADM (or, as the case may be, BDM) right-angled at D, and where OM is a perpendicular at the vertex M to the plane of the triangle. It is to be observed that, in speaking of the original pyramid as thus divided, we mean that the pyramid is the sum of the component pyramids taken each with the proper sign, + or −, as the case may be.

11. In the case of a polyhedron, this is in the like sense divisible into pyramids having for the common vertex the origin or point O, and standing on the several faces respectively; hence the polyhedron is ultimately divisible into triangular pyramids such as OADM, where ADM is a triangle right-angled at D, and where OM is a perpendicular at M to the plane of the triangle. Hence the potential of the polyhedron in regard to the point O depends upon that of the pyramid OADM; and (what is the same thing) the potential of any plane polygon in regard to the point O depends upon that of the right-angled triangle ADM, situate as above in regard to the point O. I take OM = h, MD = f, DA = g; viz., supposing, as we may do, that the plane of the triangle is parallel to that of xy, the point M on the axis of z, and the side MD parallel to the axis of x, then f, g, h will be the coordinates of the point A.



Formulæ for component triangular Pyramid, and Triangle.

12. Writing, as above, $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, $z = r \cos\theta$, and observing that h is the perpendicular distance originally called p, we have, for the potential of the pyramid,

$$V = \tfrac{1}{2}\int r^2 d\omega = \tfrac{1}{2}h^2 \int \frac{\sin\theta\, d\theta\, d\phi}{\cos^2\theta}$$
$$= \tfrac{1}{2}h^2 \int d\phi\, (\sec\theta),$$

where, ϕ being regarded as a given angle, the integral expression $\sec\theta$ must be taken from $\theta = 0$ to the value of θ corresponding to a point in the side AD. For any such point we have $f = r\sin\theta\cos\phi$, $h = r\cos\theta$, that is, $\frac{f}{h} = \tan\theta\cos\phi$, or the required value of θ is $= \tan^{-1}\frac{f}{h\cos\phi}$, and consequently that of $\sec\theta$ is $\sqrt{1+\frac{f^2}{h^2\cos^2\phi}}$, $= \frac{1}{h\cos\phi}\sqrt{f^2+h^2\cos^2\phi}$, or, as this may also be written,

$$= \frac{1}{h}\sqrt{f^2+h^2+f^2\tan^2\phi}\,;$$

hence
$$V = \tfrac{1}{2}h\int (\sqrt{f^2+h^2+f^2\tan^2\phi} - h)\, d\phi.$$

13. The first term of the integral, writing therein for a moment $\tan\phi = x$, is
$$\int \frac{(f^2+h^2+f^2x^2)\, dx}{(1+x^2)\sqrt{f^2+h^2+f^2x^2}},$$
$$= f^2\int \frac{dx}{\sqrt{f^2+h^2+f^2x^2}} + h^2\int \frac{dx}{(1+x^2)\sqrt{f^2+h^2+f^2x^2}}$$
$$= f\log(fx + \sqrt{f^2+h^2+f^2x^2}) + h\tan^{-1}\frac{hx}{\sqrt{f^2+h^2+f^2x^2}}.$$

Hence, replacing for x its value, we have

$$V = \tfrac{1}{2}h\left\{ h\tan^{-1}\frac{h\tan\phi}{\sqrt{f^2+h^2+f^2\tan^2\phi}} + f\log(f\tan\phi + \sqrt{f^2+h^2+f^2\tan^2\phi}) - h\phi \right\}$$

to be taken from $\phi = 0$ to the value of ϕ corresponding to the point A; viz., we have here $f = r\sin\theta\cos\phi$, $g = r\sin\theta\sin\phi$, $h = r\cos\theta$, and thence $\tan\phi = \frac{g}{f}$ or $f\tan\phi = g$; whence, writing for shortness, $s = \sqrt{f^2+g^2+h^2}$ (viz., s denotes the distance OA), we have

$$V = \tfrac{1}{2}h\left\{ h\tan^{-1}\frac{gh}{fs} + f\log\frac{s+g}{\sqrt{f^2+h^2}} - h\tan^{-1}\frac{g}{f} \right\};$$

or, observing that
$$\frac{s+g}{s-g} = \frac{(s+g)^2}{\sqrt{f^2+h^2}},$$

this is
$$V = \tfrac{1}{2}h\left\{ h\tan^{-1}\frac{gh}{fs} + \tfrac{1}{2}f\log\frac{s+g}{s-g} - h\tan^{-1}\frac{g}{f} \right\},$$

for the potential of the pyramid OMDA in regard to the point O; by omitting the factor $\frac{1}{2}h$, we have

$$V = h\tan^{-1}\frac{gh}{fs} + \tfrac{1}{2}f\log\frac{s+g}{s-g} - h\tan^{-1}\frac{g}{f}$$

for the potential of the triangle MDA. The expression $\tan^{-1}$ denotes, here and elsewhere, an arc included between the limits $-\frac{\pi}{2}$, $+\frac{\pi}{2}$, and which is therefore + or − according as the tangent is + or −.

Formulæ for rectangular Pyramid, and Rectangle.

14. Completing the rectangle MDAE, the potential of the triangle AME is obtained by interchanging the letters g and f; viz., we have

$$V = h\tan^{-1}\frac{fh}{gs} + \tfrac{1}{2}g\log\frac{s+f}{s-f} - h\tan^{-1}\frac{f}{g}$$

for the potential of the triangle MEA.

The sum of the two gives the potential of the rectangle MDAE; viz., for this rectangle, we have

$$V = h\left(\tan^{-1}\frac{gh}{fs} + \tan^{-1}\frac{fh}{gs} - \frac{\pi}{2}\right) + \tfrac{1}{2}f\log\frac{s+g}{s-g} + \tfrac{1}{2}g\log\frac{s+f}{s-f}.$$

But we have $\tan^{-1}\frac{gh}{fs} + \tan^{-1}\frac{hf}{gs} + \tan^{-1}\frac{fg}{hs} = \frac{\pi}{2}$;

the function on the left hand is

$$= \tan^{-1}\frac{\frac{gh}{fs} + \frac{hf}{gs} + \frac{fg}{hs} - \frac{fgh}{s^3}}{1 - \frac{f^2}{s^2} - \frac{g^2}{s^2} - \frac{h^2}{s^2}};$$

viz., the denominator being $1 - \frac{f^2+g^2+h^2}{s^2}$, $=0$, the tangent of the arc is ∞, and the component arcs being each positive and less than $\frac{\pi}{2}$, the arc in question can only be $= \frac{\pi}{2}$. We have consequently

$$V = -h\tan^{-1}\frac{fg}{hs} + \tfrac{1}{2}f\log\frac{s+g}{s-g} + \tfrac{1}{2}g\log\frac{s+f}{s-f}$$

for the potential of the rectangle MDAE. And, multiplying this by $\frac{1}{2}h$, we have $V = -\tfrac{1}{2}h^2\tan^{-1}\frac{fg}{hs} + \tfrac{1}{4}hf\log\frac{s+g}{s-g} + \tfrac{1}{4}gh\log\frac{s+f}{s-f}$

for the potential of the rectangular pyramid, vertex O and base MDAE.

Formula for the Cuboid.

15. Completing the rectangular parallelopiped, or, say for shortness, the "cuboid," the sides whereof are (f, g, h); this breaks up into three pyramids, standing on the rectangles fg, gh, and hf respectively; and the potentials for the last two pyramids are at once obtained from the last mentioned expression of V by mere cyclical interchanges of the letters. Adding the three expressions, we obtain

$$V = \tfrac{1}{2}gh \log \frac{s+f}{s-f} + \tfrac{1}{2}hf \log \frac{s+g}{s-g} + \tfrac{1}{2}fg \log \frac{s+h}{s-h} - \tfrac{1}{2}f^2 \tan^{-1} \frac{gh}{fs} - \tfrac{1}{2}g^2 \tan^{-1} \frac{hf}{gs} - \tfrac{1}{2}h^2 \tan^{-1} \frac{fg}{hs}$$

for the potential of the cuboid.

Group of Results, for Point, Line, Rectangle, and Cuboid.

16. It is convenient to prefix two results, that for the potential of the point A (mass taken to be unity) and that for the potential of the line AE (density taken to be unity, or mass of an element of length dx, taken to be $= dx$). We have, the attracted point being always at O,

Potential of point A $= \dfrac{1}{s}$ $(s = \sqrt{f^2+g^2+h^2}$, as before),

Potential of line AE $= \tfrac{1}{2} \log \dfrac{s+f}{s-f}$,

Potential of rectangle MDAE $= \tfrac{1}{2}g \log \dfrac{s+f}{s-f} + \tfrac{1}{2}f \log \dfrac{s+g}{s-g} - h \tan^{-1} \dfrac{fg}{hs}$,

Potential of cuboid $= \tfrac{1}{2}gh \log \dfrac{s+f}{s-f} + \tfrac{1}{2}hf \log \dfrac{s+g}{s-g} + \tfrac{1}{2}fg \log \dfrac{s+h}{s-h} - \tfrac{1}{2}f^2 \tan^{-1} \dfrac{gh}{fs} - \tfrac{1}{2}g^2 \tan^{-1} \dfrac{hf}{gs} - \tfrac{1}{2}h^2 \tan^{-1} \dfrac{fg}{hs}$,

which functions may be called A (f, g, h), B (f, g, h), C (f, g, h), and D (f, g, h) respectively. It is to be observed that f, g, h are taken to be each of them positive, and that s denotes in every case the positive value of $\sqrt{f^2+g^2+h^2}$; for a symmetrically situated body corresponding to negative values of each or any of these quantities, the potential has in each case its original value, without change of sign. But B is an odd function as regards f, C an odd function as regards f or g, D an odd function as regards f, g, or h; for example, C $(-f, g, \pm h)$ and C $(f, -g, \pm h)$ are each $= -$ C (f, g, h), and therefore of course C $(-f, -g, \pm h) =$ C (f, g, h).

Extension to case where the attracted point has an arbitrary position.

17. The attracted point has thus far been considered as in a definite position in regard to the attracting mass; but it is easy to pass to the

general case of any relative position whatever. Thus, for a line AB, if M be the foot of the perpendicular let fall from the point O, and if, to fix the ideas, the order of succession of the three points is A, B, M, then, with respect to the point O,

$$\text{line AB} = \text{line AM} - \text{line BM}. \qquad \underline{A \qquad B \qquad M}$$

Taking the y and z coordinates to be b, c, the x coordinates for the points A, B, M being x_0, x_1, a respectively, and in the figure ($a > x_1$, $x_1 > x_0$), then $a - x_0$, $a - x_1$ are each of them positive, $a - x_0$ being the greater, the potential of the line AM is $= \mathrm{B}(a - x_0, b, c)$, that of BM is $= \mathrm{B}(a - x_1, b, c)$, and the potential of the whole line is

$$= \mathrm{B}(a - x_0, b, c) - \mathrm{B}(a - x_1, b, c);$$

viz., this formula is proved for the case where M is situate as in the figure. But supposing that A and B retain their relative position (viz., $x_1 > x_0$), then the formula holds good for any other position of M; thus, if M be between the points A, B—viz., if the order is A, M, B—

then $\text{line AB} = \text{line AM} + \text{line BM}$,

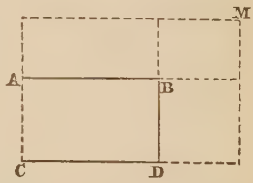
and potential is $= \mathrm{B}(a - x_0, b, c) + \mathrm{B}(x_1 - a, b, c)$,

where the second term is $= -\mathrm{B}(a - x_1, b, c)$; and so, if the order is M, A, B, then $\text{line AB} = \text{line BM} - \text{line AM}$,

and the potential is $\mathrm{B}(x_1 - a, b, c) - \mathrm{B}(x_0 - a, b, c)$, which is

$$= -\mathrm{B}(a - x_1, b, c) + \mathrm{B}(a - x_0, b, c).$$

18. Similarly for a rectangle ABCD, if M, the foot of the perpendicular from the point O, has the position shown in the figure, then



$$\begin{aligned} \text{rectangle AD} = {} & \text{rectangle MC} \\ & - \text{rectangle MA} \\ & - \text{rectangle MD} \\ & + \text{rectangle MB}, \end{aligned}$$

where O is a point on the perpendicular at the common vertex M of the four rectangles; and the resulting expression for the rectangle AD will apply to any position of the point M.

19. And in like manner for a cuboid; taking the point O in any determinate position, the cuboid may be decomposed into eight cuboids (each with the sign + or − as the case may be) having the point O for a common vertex; and the resulting expression for the potential will apply to any position whatever of the point M.

20. The results may be collected and exhibited as follows:—the coordinates of the attracted point are a, b, c; and it is assumed that

$x_1 > x_0$, $y_1 > y_0$, $z_1 > z_0$, (viz., for x the order is $+\infty$, x_1, x_0, $-\infty$, and so for y and z respectively).

Potential of point (x, y, z) is $= \mathrm{A}(a-x, b-y, c-z)$

Potential of line (x_1, y, z), (x_0, y, z) is $= \mathrm{B}(a-x_0, b-y, c-z) - \mathrm{B}(a-x_1, b-y, c-z)$;

Potential of rectangle (x_1, y_1, z), (x_0, y_1, z), (x_1, y_0, z), (x_0, y_0, z) is $= \mathrm{C}(a-x_0, b-y_0, c-z) - \mathrm{C}(a-x_0, b-y_1, c-z) - \mathrm{C}(a-x_1, b-y_0, c-z) + \mathrm{C}(a-x_1, b-y_1, c-z)$;

Potential of cuboid (x_1, y_1, z_1), (x_0, y_1, z_1), (x_1, y_0, z_1), (x_0, y_0, z_1), (x_1, y_1, z_0), (x_0, y_1, z_0), (x_1, y_0, z_0), (x_0, y_0, z_0) is $= \mathrm{D}(a-x_0, b-y_0, c-z_0) - \mathrm{D}(a-x_1, b-y_0, c-z_0) - \mathrm{D}(a-x_0, b-y_1, c-z_0) + \mathrm{D}(a-x_1, b-y_1, c-z_0) - \mathrm{D}(a-x_0, b-y_0, c-z_1) + \mathrm{D}(a-x_1, b-y_0, c-z_1) + \mathrm{D}(a-x_0, b-y_1, c-z_1) - \mathrm{D}(a-x_1, b-y_1, c-z_1)$.

21. These are connected together as follows, viz.:—

$$\text{Potential of line} = \int_{x_0}^{x_1} dx \text{ Potential of point},$$

$$\text{Potential of rectangle} = \int_{y_0}^{y_1} dy \text{ Potential of line},$$

$$\text{Potential of cuboid} = \int_{z_0}^{z_1} dz \text{ Potential of rectangle},$$

equations which are in fact of the form

$$\mathrm{B}(x, y, z) = \int dx\, \mathrm{A}(x, y, z),$$

$$\mathrm{C}(x, y, z) = \int dy\, \mathrm{B}(x, y, z),$$

$$\mathrm{D}(x, y, z) = \int dz\, \mathrm{C}(x, y, z).$$

Differential properties of the functions A, B, C, D.

22. These relations, with other allied ones, may be verified as follows. Writing $r = \sqrt{x^2+y^2+z^2}$, the fundamental forms are

$$\log \frac{r+x}{r-x}, \quad \text{and} \quad \tan^{-1} \frac{yz}{rx},$$

and we have $d_x r = \frac{x}{r}$, &c., and thence

$$d_x \log \frac{r+x}{r-x} = \frac{1+\frac{x}{r}}{r+x} - \frac{-1+\frac{x}{r}}{r-x}, = \frac{1}{r} + \frac{1}{r}, = \frac{2}{r},$$

$$d_y \log \frac{r+x}{r-x} = \frac{\frac{y}{r}}{r+x} - \frac{\frac{y}{r}}{r-x}, = \frac{y}{r}\left(\frac{1}{r+x} - \frac{1}{r-x}\right), = \frac{-2xy}{r(r^2-x^2)},$$

&c. &c. &c.

$$d_x \tan^{-1}\frac{yz}{rx} = \frac{-yz\left(r+\frac{x^2}{r}\right)}{r^2x^2+y^2z^2} = \frac{-yz}{r}\frac{r^2+x^2}{r^2x^2+y^2z^2};$$

or, since $r^2+x^2 = (r^2-y^2)+(r^2-z^2)$ and $r^2x^2+y^2z^2 = (r^2-y^2)(r^2-z^2)$,

this is $$= -\frac{yz}{r}\left(\frac{1}{r^2-y^2} + \frac{1}{r^2-z^2}\right),$$

$$d_y \tan^{-1}\frac{yz}{rx} = \frac{rx\,.\,z - yz\,.\,\frac{xy}{r}}{r^2x^2+y^2z^2} = \frac{xz}{r}\frac{r^2-y^2}{r^2x^2+y^2z^2},$$

which, the denominator being, as before, $(r^2-y^2)(r^2-z^2)$, is

$$= \frac{xz}{r}\frac{1}{r^2-z^2}.$$

It is now easy to form the following results:—

23. First, $u = \mathrm{A}(x, y, z) = \frac{1}{r}$ (symmetrical),

$$d_x u = -\frac{x}{r^3}, \text{ &c.},$$

$$d_x^2 u = \frac{3x^2}{r^5} - \frac{1}{r^3}, \text{ &c.}, \quad d_x d_y u = \frac{3xy}{r^5}, \text{ &c.},$$

and thence $$(d_x^2 + d_y^2 + d_z^2)u = 0.$$

24. Secondly,

$u = \mathrm{B}(x, y, z) = \frac{1}{2}\log\frac{r+x}{r-x}$ (symmetrical as to y, z);

then

$$d_x u = \frac{1}{r}\left(= \mathrm{A}(x, y, z)\right), \quad d_y u = \frac{-xy}{r(r^2-x^2)}, \text{ &c.},$$

$$d_x^2 u = -\frac{x}{r^3}, \quad d_x d_y u = -\frac{y}{r^3}, \quad d_y d_z u = \frac{xyz}{r^3(r^2-x^2)} + \frac{2xyz}{r(r^2-x^2)^2}, \text{ &c.},$$

$$d_y^2 u = \frac{-x}{r(r^2-x^2)} + \frac{2xy^2}{r(r^2-x^2)^2} + \frac{xy^2}{r^3(r^2-x^2)}, \text{ &c.};$$

and thence

$$(d_x^2 + d_y^2 + d_z^2)u = -\frac{x}{r^3} - \frac{2x}{r(r^2-x^2)} + \frac{2x}{r(r^2-x^2)} + \frac{x}{r^3}, = 0.$$

25. Thirdly,

$$u = \mathrm{C}(x, y, z) = \tfrac{1}{2} y \log \frac{r+x}{r-x} + \tfrac{1}{2} x \log \frac{r+y}{r-y} - z \tan^{-1} \frac{xy}{zr}$$

(symmetrical as to x, y),

$$d_y u = \tfrac{1}{2} \log \frac{r+x}{r-x} \left(= \mathrm{B}(x, y, z) \right),$$

in verification whereof, observe that the remaining terms are

$$= -\frac{xy^2}{r \,.\, r^2 - x^2} + \frac{x}{r} - \frac{xz^2}{r} \frac{r^2 - y^2}{(r^2 - x^2)(r^2 - y^2)},$$

$$= \frac{x}{r} \left(-\frac{y^2}{r^2 - x^2} + 1 - \frac{z^2}{r^2 - x^2} \right),$$

$$= \frac{x}{r(r^2 - x^2)} (-y^2 + r^2 - x^2 - z^2),$$

which is $= 0$.

$$d_z u = -\frac{xyz}{r(r^2 - x^2)} - \frac{xyz}{r(r^2 - y^2)} + \frac{xyz}{r} \frac{r^2 - x^2 + r^2 - y^2}{(r^2 - x^2)(r^2 - y^2)} - \tan^{-1} \frac{xy}{zr},$$

$$= -\tan^{-1} \frac{xy}{zr},$$

$$d_x^2 u = -\frac{xy}{r(r^2 - y^2)},$$

$$d_z^2 u = \frac{xy}{r} \frac{r^2 - x^2 + r^2 - y^2}{(r^2 - x^2)(r^2 - y^2)}, = \frac{xy}{r} \left(\frac{1}{r^2 - x^2} + \frac{1}{r^2 - y^2} \right),$$

and thence, $$(d_x^2 + d_y^2 + d_z^2) u = 0.$$

26. Fourthly,

$$u = \mathrm{D}(x, y, z) = \tfrac{1}{2} yz \log \frac{r+x}{r-x} + \tfrac{1}{2} zx \log \frac{r+y}{r-y} + \tfrac{1}{2} xy \log \frac{r+z}{r-z}$$

$$- \tfrac{1}{2} x^2 \tan^{-1} \frac{yz}{xr} - \tfrac{1}{2} y^2 \tan^{-1} \frac{zx}{ry} - \tfrac{1}{2} z^2 \tan^{-1} \frac{xy}{zr} \text{ (symmetrical)},$$

$$d_z u = \tfrac{1}{2} y \log \frac{r+x}{r-x} + \tfrac{1}{2} x \log \frac{r+y}{r-y} - z \tan^{-1} \frac{xy}{zr} = \mathrm{C}(x, y, z),$$

$$d_z^2 u = -\tan^{-1} \frac{xy}{zr};$$

and thence

$$(d_x^2 + d_y^2 + d_z^2) u = -\tan^{-1} \frac{yz}{xr} - \tan^{-1} \frac{zx}{yr} - \tan^{-1} \frac{xy}{zr}$$

$$= -\tan^{-1} \frac{\frac{yz}{xr} + \frac{zx}{yr} + \frac{xy}{zr} - \frac{xyz}{r^3}}{1 - \frac{x^2}{r^2} - \frac{y^2}{r^2} - \frac{z^2}{r^2}};$$

viz., the denominator being $= 0$, the arc is $\pm \frac{\pi}{2}$, or we have

$$(d_x^2 + d_y^2 + d_z^2) u = \mp \frac{\pi}{2},$$

the value being $-\frac{\pi}{2}$ if x, y, z are all three of them, or only one, positive; but $+\frac{\pi}{2}$ if they are all three of them, or only one, negative.

Application to the Potentials of the Point, the Line, the Rectangle, and the Cuboid.

27. Take now V to denote in succession the foregoing expressions of the potential of a point, a line, a rectangle, or a cuboid, on the point (a, b, c). In the first three cases respectively, each of the component terms is reduced to zero by the operator $d_a^2 + d_b^2 + d_c^2$; and we have, therefore,

$$(d_a^2 + d_b^2 + d_c^2)\,\mathrm{V} = 0,$$

which is as it should be. But in the case of the cuboid, each of the eight component terms is by the operator reduced to $\mp\frac{\pi}{2}$, and we have therefore

$$(d_a^2 + d_b^2 + d_c^2)\,\mathrm{V} = \Sigma\left\{\pm\left(\mp\frac{\pi}{2}\right)\right\};$$

Σ denoting the sum of eight terms, the $\pm$ denoting $+$ or $-$, according to the sign of the term in the formula (viz., in four cases this is $+$, and in four cases it is $-$), and the $\mp\frac{\pi}{2}$ the value $\frac{\pi}{2}$ with its proper sign depending on the signs of the quantities $(a-x_0, b-y_0, c-z_0)$, &c., as explained in the preceding Number.

Suppose for a moment $a>x_1$, $b>y_1$, $c>z_1$, or the attracted point in one of the regions exterior to the cuboid; then $\mp\frac{\pi}{2}$ will in each case be $=-\frac{\pi}{2}$, and the sign $\mp$ being $+$ for four of the terms and $-$ for the four remaining terms, the sum is $=0$. And similarly, in all cases where the attracted point is exterior to the cuboid, the sum of the eight terms is $=0$. But when the attracted point is interior, that is, when $a>x_0<x_1$, $b>y_0<y_1$, $c>z_0<z_1$, then it is found that, for the four terms which have the sign $+$, the value of $\mp\frac{\pi}{2}$ is $=-\frac{\pi}{2}$; and for the four terms which have the sign $-$, its value is $=+\frac{\pi}{2}$; whence, in the sum, each term is $=-\frac{\pi}{2}$, or the value is $=-4\pi$. Hence, in the case of the cuboid, we have

$$(d_a^2 + d_b^2 + d_c^2)\,\mathrm{V} = 0 \text{ or } -4\pi,$$

according as the attracted point is external or internal.

Verification in regard to the Rectangle.

28. I start from the formula

$$\begin{aligned} V = \quad & C(a-x_0,\ b-y_0,\ c) \\ & - C(a-x_1,\ b-y_0,\ c) \\ & - C(a-x_0,\ b-y_1,\ c) \\ & + C(a-x_1,\ b-y_1,\ c), \end{aligned}$$

where, as before, $x_1 > x_0$, $y_1 > y_0$. V is here a function of (a, b, c), satisfying the partial differential equation

$$(d_a^2 + d_b^2 + d_c^2)\, V = 0,$$

and (as is easily verified) vanishing when any one of the variables a, b, c becomes infinite, and which does not become infinite for any values of a or b, or any positive value of c. Hence, by a theorem of Green's,* there exists on the plane $z = 0$ a distribution of matter giving rise to the potential V; and not only so, but the density at any point (x, y) of the plane is given by the formula

$$\rho = -\frac{1}{2\pi}\left(\frac{dW}{dc}\right)_{c=0},$$

where W is what V becomes on writing therein x, y in place of a, b, and $c = 0$ is regarded as an indefinitely small positive quantity.

We have $d_c C(x, y, c) = -\tan^{-1}\dfrac{xy}{cr}$, where $r = \sqrt{x^2+y^2+c^2}$.

And hence

$$\begin{aligned} d_c W = & -\tan^{-1}\frac{(x-x_0)(y-y_0)}{c\sqrt{(x-x_0)^2+(y-y_0)^2+c^2}} \\ & +\tan^{-1}\frac{(x-x_1)(y-y_0)}{c\sqrt{(x-x_1)^2+(y-y_0)^2+c^2}} \\ & +\tan^{-1}\frac{(x-x_0)(y-y_1)}{c\sqrt{(x-x_0)^2+(y-y_1)^2+c^2}} \\ & -\tan^{-1}\frac{(x-x_1)(y-y_1)}{c\sqrt{(x-x_1)^2+(y-y_1)^2+c^2}}. \end{aligned}$$

Putting $c = 0$, as above, each arc is $= \dfrac{\pi}{2}$ or $-\dfrac{\pi}{2}$, according as the fraction under the $\tan^{-1}$ is positive or negative—that is, according as the numerator is positive or negative. Suppose for a

* The theorem in question is a particular case of Green's, $4\pi\rho = -\left(\dfrac{dV}{dw} + \dfrac{dV'}{dw}\right)$ ["Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism" (1828), see p. 31 of the Collected Works]; viz., the surface is here a plane, and $V = V'$. And it is also a particular case of the formula $\rho' = \dfrac{-\Gamma\frac{1}{2}(n-1)}{2\pi^{\frac{1}{2}s}\Gamma\frac{1}{2}(n-s+1)}\, P'$ ["Memoir on the Determination of the Exterior and Interior Attraction of Ellipsoids of Variable Densities" (1835), see p. 199 of the Collected Works]; viz., s is taken $=2$; and Green's extra-spatial coordinate u then becomes the coordinate z of ordinary tri-dimensional space.

moment $x > x_1$, $y > y_1$, viz., the point (x, y), is here in a region exterior to the rectangle $(x_1, y_1), (x_1, y_0), (x_0, y_1), (x_0, y_0)$: the value of $d_c W$ is $= -\frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2} - \frac{\pi}{2}, = 0$; and similarly, for every other position of the point (x, y) exterior to the rectangle, the value is $= 0$. But for a point interior to the rectangle, we have $x < x_1 > x_0$, $y < y_1 > y_0$, and in this case the value is $-\frac{\pi}{2} + \left(-\frac{\pi}{2}\right) + \left(-\frac{\pi}{2}\right) - \frac{\pi}{2}, = -2\pi$.

Hence $$\rho, = -\frac{1}{2\pi}\left(d_c W\right)_{c=0}, \text{ is } = 0 \text{ or } 1,$$

according as the point is exterior or interior to the rectangle, viz., the distribution producing the potential in question is a uniform distribution (density unity) over the rectangle, which is as it should be.

Potential of a Cuboidal Surface.

29. The preceding formulæ lead to the expression of the potential of a cuboidal surface (viz., the surface composed of the six faces of a cuboid, each of them being considered as a plate of the same uniform density) upon a point a, b, c. Writing, for convenience,

$$\mathrm{E}(f, g, h) = \tfrac{1}{2}(g+h)\log\frac{s+f}{s-f} + \tfrac{1}{2}(h+f)\log\frac{s+g}{s-g} + \tfrac{1}{2}(f+g)\log\left(\frac{s+h}{s-h}\right) - f\tan^{-1}\frac{gh}{fs} - g\tan^{-1}\frac{hf}{gs} - h\tan^{-1}\frac{fg}{hs},$$

where each term is supposed to have (compounded with its expressed sign) a sign $\pm$, as follows: viz., in any fg term $\left(\frac{1}{2}f\log\frac{s+g}{s-g}, \frac{1}{2}g\log\frac{s+f}{s-f}, \text{ or } h\tan^{-1}\frac{fg}{hs}\right)$, this sign $\pm$ is $+$ if f and g are both positive or both negative, but is $-$ if f and g are the one of them positive and the other negative; and the like as to the gh terms and the hf terms respectively. And this being so, the expression for the potential (applying as well to an interior as to an exterior point) is

$$\begin{aligned} \mathrm{V} = \quad & \mathrm{E}(a-x_0, b-y_0, c-z_0) \\ & + \mathrm{E}(a-x_1, b-y_0, c-z_0) \\ & + \mathrm{E}(a-x_0, b-y_1, c-z_0) \\ & + \mathrm{E}(a-x_0, b-y_0, c-z_0) \\ & + \mathrm{E}(a-x_0, b-y_0, c-z_1) \\ & + \mathrm{E}(a-x_1, b-y_0, c-z_1) \\ & + \mathrm{E}(a-x_0, b-y_1, c-z_1) \\ & + \mathrm{E}(a-x_1, b-y_1, c-z_1). \end{aligned}$$

It is, in fact, easy to verify that the final result, interpreted as above, represents the sum of the six positive values, which are the values of the potential for the six faces of the cuboid respectively.

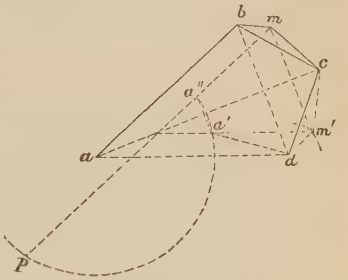
The following are the two letters from M. Mannheim to Prof. Sylvester :—

Paris, 28 *Octobre* 1874.

Mon cher Monsieur Sylvester,

Voici une démonstration géométrique de cette jolie proposition que vous m'avez énoncée :

"*abcd* est un quadrilatère dont les côtés sont de grandeurs invariables et tel que *ab* égal *ad* et *bc* égal *cd*. Les points *a* et *d* étant fixes, on déforme ce quadrilatère : un point *m* lié invariablement au côté *bc* décrit une podaire de conique."



Sur le côté *cd*, qui est égal à *bc*, je construis le triangle *cdm′* égal au triangle *bmc*, de façon que le point *m′* soit le symétrique de *m* par rapport à la diagonale *ac*. Sur le côté *dm′* de ce triangle je construis le triangle *a′dm′* semblable au triangle *adc*.

Il résulte de cette construction que, quelle que soit la situation des côtés du quadrilatère *abcd*, le point *a′* ainsi construit sera toujours le même, et que le côté *a′m′* fera toujours un angle constant avec son homologue *ac*.

Par suite l'angle *a′m′m*, complémentaire de cet angle, sera aussi constant.

Mais le côté *a′m′* de ce dernier angle, pendant la déformation du quadrilatère, passe toujours par le point fixe *a′*, le sommet *m′* de cet angle reste sur la circonférence décrite par ce point, donc l'autre côté *mm′* enveloppe une conique.

Prenons, par rapport à *ac*, la droite *a″m* symétrique de *a′m′* : cette droite fait un angle constant avec *a′a″* qui est perpendiculaire à *ac*. Mais le point *a″* est sur la circonférence décrite du point *a* comme centre avec *aa′* pour rayon, donc le point *p*, où le côté *a″m* de l'angle constant *a′a″p* rencontre cette circonférence, est un point fixe. La droite *pm* est donc issue d'un point fixe *p*, et comme elle fait un angle constant avec la tangente *mm′* à une conique, le point *m* décrit une podaire de conique.

MANNHEIM.

Paris, 4 *Nov.* 1874.

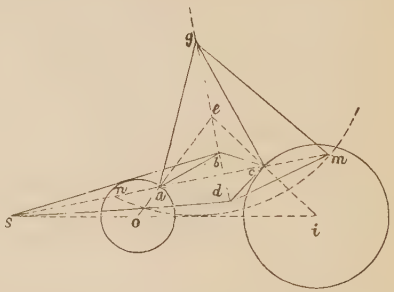
Mon cher Monsieur Sylvester,

Je continue à vous écrire les quelques remarques que j'ai faites sur les systèmes articulés.

abcd est le losange de Peaucellier ; les sommets *b*, *d* sont liés au point fixe *s*, et le sommet *a* au point fixe *o* ; le sommet libre *c*

décrit la circonférence *i*. Cette circonférence et la circonférence *o* ont pour centre de similitude le point *s*.

Sur la diagonale *bd* prenons le point *g* et réunissons ce point au moyen des tiges *ga*, *gc* aux sommets *a* et *c*. On a $sg^2-gc^2=sb^2-bc^2$. Il résulte de là que pendant le déplacement du losange *abcd* le point *g* décrit une circonférence dont le centre est *s*. D'après cela au losange de Peaucellier, on peut substituer le quadrilatère *gadc* dont les sommets *a*, *g*, *d* sont liés aux points fixes *o*, *s*.



Du point *g* comme centre avec une longueur arbitraire décrivons une circonférence qui rencontre la droite *sc* aux points *m*, *n*. Joignons le point *m* au point *d*: la distance *dm* reste constante, puisque *sg*, *sd*, *gm* sont des longueurs constantes.

Considérons maintenant le quadrilatère articulé *agmd*, dont les côtés sont inégaux et qui n'est assujetti qu'à la condition d'avoir ses diagonales perpendiculaires entre elles. Je vais démontrer que, si les sommets *g*, *d* sont liés au point *s* et le sommet *a* au point *o*, le sommet libre *m* décrira une anallagmatique du 4e ordre (bicircular quartic).

La circonférence décrite du point *g* comme centre avec *gm* pour rayon coupe à angle droit la circonférence dont le centre est *s* et le rayon $\sqrt{sg^2-gm^2}$. [Cette dernière circonférence je la désignerai par (*s*).]

Toutes les circonférences passant par *m* et *n* coupent à angle droit la circonférence (*s*).

En particulier, nous pouvons prendre la circonférence dont le centre est en *e* rencontre des rayons *oa*, *ic*. Pendant le déplacement du losange ce point *e* décrit une conique dont la tangente en *e* est la droite *ed*. Nous avons donc une circonférence dont le centre est en *e* sur une conique, et qui coupe orthogonalement une circonférence (*s*); toutes les circonférences ainsi construites enveloppent une anallagmatique du 4e ordre. Les points où la circonférence, dont le centre est *e*, touche son enveloppe, s'obtiennent en prenant les points où elle est coupée par la perpendiculaire abaissée du point *s* sur la tangente *ed* à la conique (*e*). Ces points de rencontre sont *m* et *n*. Je puis donc dire: "Lorsque le quadrilatère *agmd*, dont les côtés sont inégaux, mais dont les diagonales sont perpendiculaires entre elles, se déforment de façon que le sommet *a* décrive une circonférence (*o*) et les sommets *g*, *d* des circonférences ayant même centre *s* sur la diagonale *ma*, le sommet libre *m* décrira une anallagmatique du 4e ordre."

Veuillez agréer l'assurance de mes sentiments affectueux,

MANNHEIM.

January 14th, 1875.

Prof. H. J. S. Smith, F.R.S., President, in the Chair.

The following gentlemen were elected Ordinary Members :—Messrs. J. W. Russell, C. M. Leudesdorf, E. B. Elliott, H. M. Jeffery, C. Smith, and B. Williamson ; and Drs. Klein, Kronecker, and Zeuthen, Honorary Foreign Members.

Mr. J. Hammond, B.A., was admitted into the Society, and Mr. C. E. Bickmore, M.A., Fellow of New College, Oxford, was proposed for election. M. Camille Jordan and Major J. R. Campbell were among the visitors present.

Mr. Glaisher gave an abstract of Prof. Cayley's two papers "On the Potentials of an Ellipse and Circle," and "On the Attraction of an Ellipsoidal Shell." Mr. J. Hammond read his paper "On the Solution of Linear Differential Equations in Series." Major Campbell explained the construction and use of his Mechanical Calculator, and presented two of his instruments to the Society. Prof. Sylvester made a brief communication "On the representation of any Unicursal Curve and its Nodes, in terms of the parametric coefficients, and on Roberts' cases of Unicursal Three-bar Motion." M. Camille Jordan spoke on the subject of this communication. Prof. Henrici exhibited a model of a Hyperboloid.

The following presents were received :—

Two Mechanical Calculators : from Major J. R Campbell.

"On Tautochronous and Brachystochronous Curves for Parallel and Concurrent Forces," Quarterly Journal, No. 47, 1873. A continuation of the same, No. 49, 1873. And "On the free equilibrium of a uniform cord compared with the free motion of a material particle under the action of a central force," Quarterly Journal, No. 51 : from the Author, the Rev. Prof. Townsend, F.R.S.

"Sitzungsberichte der physikalisch-medicinischer Societät zu Erlangen," 6 Heft, Nov. 1873 bis Aug. 1874.

"Crelle," 79 Band, Drittes Heft, Berlin, 1874.

"Bulletin de la Société Mathématique," Oct. 1874, Paris.

"Remarks on the great Logarithmic and Trigonometrical Tables, computed in the Bureau du Cadastre, under the direction of M. Prony," by Edward Sang, Dec. 1874.

Specimen pages of a Table of the Logarithms of all numbers up to One million, in preparation, by Edward Sang : from the Author.

"The Cone and its Sections treated Geometrically," by S. A. Renshaw, 1875 : from the Author.

Carte-de-visite likeness of Mr. C. J. C. Price, of Exeter College, Oxford.

On the Potential of the Ellipse and the Circle. *By* Prof. Cayley.

[*Read January 14th*, 1875.]

The Potential of the Ellipse.

1. I consider the potential of an ellipse (or say an elliptic plate of uniform density); viz., this is

$$V = \int \frac{dx\,dy}{\sqrt{(a-x)^2+(b-y)^2+c^2}},$$

the limits being given by the equation $\frac{x^2}{f^2}+\frac{y^2}{g^2}=1$.

Writing herein $x=mf\cos u$, $y=mg\sin u$, we have $dx\,dy=fg\,m\,dm\,du$; and consequently

$$V = fg\int \frac{m\,dm\,du}{\sqrt{(a-mf\cos u)^2+(b-mg\sin u)^2+c^2}},$$

where the integrations are to be taken from $m=0$ to $m=1$, and from $u=0$ to $u=2\pi$.

2. It is to be remarked that, by first performing the integration in regard to m, we may reduce the potential to the form $\int du.F$, where F is an *algebraic* function of $\cos u$, $\sin u$; and that the result so obtained, although in the general case too complex to be manageable, is a useful one in the case $f=g$, where the ellipse becomes a circle. The case of the circle will be treated of separately, but in the general case it will be sufficient to show that the integral is of the form in question.

3. To accomplish this, writing

$$A = a^2+b^2+c^2,$$
$$B = af\cos u+bg\sin u,$$
$$C = f^2\cos^2 u+g^2\sin^2 u,$$

then the integral in regard to m is

$$\int \frac{m\,dm}{\sqrt{A-2Bm+Cm^2}},$$

which is

$$= \frac{1}{C}\sqrt{A-2Bm+Cm^2}+\frac{B}{C\sqrt{C}}\log\left\{Cm-B+\sqrt{C}\,\sqrt{A-2Bm+Cm^2}\right\};$$

or, taken between the limits 0, 1, this is

$$= \frac{1}{C}\left(\sqrt{A-2B+C}-\sqrt{A}\right)+\frac{B}{C\sqrt{C}}\log\left\{\frac{C-B+\sqrt{C}\,\sqrt{A-2B+C}}{-B+\sqrt{C}\,\sqrt{A}}\right\};$$

and we have therefore

$$V = fg\int du\,\frac{1}{C}\,(\sqrt{A-2B+C}-\sqrt{A}) + fg\int du\,\frac{(af\cos u+bg\sin u)}{(f^2\cos^2 u+g^2\sin^2 u)^{\frac{3}{2}}}\log\Upsilon,$$

where, for greater clearness, the value of the coefficient $\frac{B}{C\sqrt{C}}$ of the logarithmic term has been written at full length.

4. But this coefficient admits of algebraic integration, viz., we have

$$fg\int du\,\frac{af\cos u+bg\sin u}{(f^2\cos^2 u+g^2\sin^2 u)^{\frac{3}{2}}} = \frac{ag\sin u-bf\cos u}{(f^2\cos^2 u+g^2\sin^2 u)^{\frac{1}{2}}};$$

hence, integrating the second term by parts, we have

$$V = fg\int du\,\frac{1}{C}\,\{\sqrt{A-2B+C}-\sqrt{A}\}$$
$$+\frac{ag\sin u-bf\cos u}{(f^2\cos^2 u+g^2\sin^2 u)^{\frac{1}{2}}}\log\Upsilon$$
$$-\int du\,\frac{ag\sin u-bf\cos u}{f^2\cos^2 u+g^2\sin^2 u}\cdot\frac{\Upsilon'}{\Upsilon},$$

where the second term, taken between the limits $u=0$, $u=2\pi$, is $=0$; and. $\frac{\Upsilon'}{\Upsilon}$ being an algebraic function of $\sin u$, $\cos u$, the potential is expressed in the form in question.

5. But we may, by means of a transformation upon u (that made use of in Gauss' Memoir on the attraction of an elliptic ring), transform the expression so as to obtain the integral in regard to m under a much more simple form. We, in fact, assume

$$\cos u = \frac{\alpha+\alpha'\cos T+\alpha''\sin T}{\gamma+\gamma'\cos T+\gamma''\sin T},$$

$$\sin u = \frac{\beta+\beta'\cos T+\beta''\sin T}{\gamma+\gamma'\cos T+\gamma''\sin T},$$

where the nine coefficients are such that identically

$$(\alpha+\alpha'\cos T+\alpha''\sin T)^2+(\beta+\beta'\cos T+\beta''\sin T)^2-(\gamma+\gamma'\cos T+\gamma''\sin T)^2$$
$$=\cos^2 T+\sin^2 T-1$$

(this of course renders the two equations consistent); and also that

$$(a-mf\cos u)^2+(b-mg\sin u)^2+c^2$$
$$=\frac{1}{(\gamma+\gamma'\cos T+\gamma''\sin T)^2}\,(G+G'\cos^2 T+G''\sin^2 T).$$

This last condition gives, for the determination of the coefficients G, G′, G″, the identity

$$(\theta-G)(\theta+G')(\theta+G'') = -(\theta+m^2f^2)(\theta+m^2g^2)\theta\left\{\frac{a^2}{\theta+m^2f^2}+\frac{b^2}{\theta+m^2g^2}+\frac{c^2}{\theta}-1\right\},$$

or, what is the same thing, G, $-G'$, $-G''$ are the roots of the equation

$$\frac{a^2}{\theta+m^2f^2}+\frac{b^2}{\theta+m^2g^2}+\frac{c^2}{\theta}-1=0;$$

the equation has one positive root, which may be taken to be G, and two negative roots, which will then be $-G'$, $-G''$; viz., G, G′, G″ are thus all positive; and G denotes the positive root of the last mentioned equation.

6. We have $$du = \frac{dT}{(G+G'\cos T+G''\sin T)^2},$$

and thence $$V = fg\int m\,dm\int\frac{dT}{(G+G'\cos^2 T+G''\sin^2 T)^{\frac{1}{2}}},$$

the integral in regard to T being taken from 0 to 2π; or, what is the same thing, we may multiply by 4 and take the integral only from 0 to $\frac{\pi}{2}$; viz., we thus have

$$V = 4fg\int m\,dm\int_0^{\frac{1}{2}\pi}\frac{dT}{(G+G'\cos^2 T+G''\sin^2 T)^{\frac{1}{2}}},$$

where the integral in regard to T can be at once reduced to the standard form of an elliptic function, or it might be calculated by Gauss' method of the arithmetico-geometrical mean.

7. But, for the present purpose, a further reduction is required. Writing $t = G+(G+G')\cot^2 T$, we have

$$t-G = (G+G')\frac{\cos^2 T}{\sin^2 T},$$

$$t+G' = (G+G')\frac{1}{\sin^2 T},$$

$$t+G'' = (G+G'\cos^2 T+G''\sin^2 T)\frac{1}{\sin^2 T};$$

whence $$\sqrt{t-G\,.\,t+G'\,.\,t+G''} = (G+G')(G+G'\cos^2 T+G''\sin^2 T)^{\frac{1}{2}}\frac{\cos T}{\sin^3 T},$$

and moreover $$dt = -2(G+G')\frac{\cos T}{\sin^3 T}dT.$$

Hence $$\frac{dt}{\sqrt{t-G\,.\,t+G'\,.\,t+G''}} = \frac{-2\,dT}{\{G+G'\cos^2 T+G''\sin^2 T\}^{\frac{1}{2}}};$$

and observing that to the limits $0, \frac{\pi}{2}$ of T correspond the limits ∞, G of t, we thence obtain

$$V = 2fg \int m\,dm \int_G^\infty \frac{dt}{\sqrt{t-G \,.\, t+G' \,.\, t+G''}};$$

or, what is the same thing,

$$V = 2fg \int m\,dm \int_G^\infty \frac{dt}{\sqrt{t\,(t+m^2f^2)\,(t+m^2g^2)\left(1-\frac{a^2}{t+m^2f^2}-\frac{b^2}{t+m^2g^2}-\frac{c^2}{t}\right)}},$$

where G denotes, as before, the positive root of the equation

$$\frac{a^2}{\theta+m^2f^2}+\frac{b^2}{\theta+m^2g^2}+\frac{c}{\theta}-1=0.$$

8. Writing for t, m^2t, and for G, m^2G, the formula becomes

$$V = 2fg \int m\,dm \int_G^\infty \frac{dt}{\sqrt{t \,.\, t+f^2 \,.\, t+g^2\left(m^2-\frac{a^2}{t+f^2}-\frac{b^2}{t+g^2}-\frac{c^2}{t}\right)}},$$

where G now denotes the positive root of the equation

$$\frac{a^2}{\theta+f^2}+\frac{b^2}{\theta+g^2}+\frac{c^2}{\theta}-m^2=0.$$

G is thus a function of m, but it is to be remarked that the integration in respect to m can be performed through the integral sign $\int_G^\infty dt$ in precisely the same way as if G were constant; and that we, in fact, have $V = 2fg\left[\int_G^\infty dt \sqrt{m^2-\frac{a^2}{t+f^2}-\frac{b^2}{t+g^2}-\frac{c^2}{t}}\;\frac{1}{\sqrt{t \,.\, t+f^2 \,.\, t+g^2}}\right]$, where the function of m is to be taken between the limits 0, 1. The reason is that, differentiating this last integral in respect to m, the term depending on the variation of the limit G is

$$\sqrt{m^2-\frac{a^2}{G+f^2}-\frac{b^2}{G+g^2}-\frac{c^2}{G}}\;\frac{1}{\sqrt{G \,.\, G+f^2 \,.\, G+g^2}}\frac{dG}{dm},$$

which is $= 0$ in virtue of the equation which defines G; hence the whole result is the term arising from the variation of m in so far as it appears explicitly.

9. Proceeding next to take the function of m between the two limits for $m = 0$ we have $G = \infty$, and the integral vanishes; for $m = 1$ we have G the positive root of the equation

$$\frac{a^2}{\theta+f^2}+\frac{b^2}{\theta+g^2}+\frac{c^2}{\theta}-1=0;$$

or, using θ to denote the positive root of this equation, the value is $G = \theta$; we thus finally obtain

$$V = 2fg \int_\theta^\infty dt \sqrt{1 - \frac{a^2}{t+f^2} - \frac{b^2}{t+g^2} - \frac{c^2}{t}} \quad \frac{1}{\sqrt{t \,.\, t+f^2 \,.\, t+g^2}}$$

as the expression for the potential of the ellipse semiaxes (f, g) on the point (a, b, c).

Case where the Attracted Point is on the Focal Hyperbola.

10. The result becomes very simple when the attracted point is in the focal hyperbola of the ellipse, viz., when we have $b = 0$ and $\frac{a^2}{f^2-g^2} - \frac{c^2}{g^2} = 1$. The function $1 - \frac{a^2}{t+f^2} - \frac{b^2}{t+g^2} - \frac{c^2}{t}$ is here

$$\begin{aligned}
&= \frac{a^2}{f^2-g^2} - \frac{c^2}{g^2} - \frac{a^2}{f^2+t} - \frac{c^2}{t} \\
&= (t+g^2)\left\{ \frac{a^2}{(t+f^2)(f^2-g^2)} - \frac{c^2}{g^2 t} \right\} \\
&= (t+g^2)\left\{ \left(1 + \frac{c^2}{g^2}\right) \frac{1}{t+f^2} - \frac{c^2}{g^2 t} \right\} \\
&= \frac{t+g^2}{t(t+f^2)} \left(t - \frac{c^2 f^2}{g^2} \right).
\end{aligned}$$

Hence also $\theta = \frac{c^2 f^2}{g^2}$, or, introducing this value, the function in question becomes $= \frac{(t+g^2)(t-\theta)}{t(t+f^2)}$,

and we have

$$\begin{aligned}
V &= 2fg \int_\theta^\infty dt \, \frac{\sqrt{t+g^2 \,.\, t-\theta}}{\sqrt{t \,.\, t+f^2}} \quad \frac{1}{\sqrt{t \,.\, t+f^2 \,.\, t+g^2}} \\
&= 2fg \int_\theta^\infty \frac{dt \sqrt{t-\theta}}{t \,.\, t+f^2},
\end{aligned}$$

which, writing $t = x^2 + \theta$, becomes

$$\begin{aligned}
V &= 4fg \int_0^\infty \frac{x^2 dx}{x^2+\theta \,.\, x^2+\theta+f^2} \\
&= \frac{4fg}{f^2} \int_0^\infty \left(\frac{\theta+f^2}{x^2+\theta+f^2} - \frac{\theta}{x^2+\theta} \right) dx \\
&= \frac{4g}{f} \left(\sqrt{\theta+f^2} \tan^{-1} \frac{x}{\sqrt{\theta+f^2}} - \sqrt{\theta} \tan^{-1} \frac{x}{\sqrt{\theta}} \right)_0^\infty \\
&= 2\pi \frac{g}{f} \left(\sqrt{\theta+f^2} - \sqrt{\theta} \right);
\end{aligned}$$

or, substituting for θ its value $\frac{c^2 f^2}{g^2}$, this is

$$V = 2\pi \left(\sqrt{c^2+g^2} - c \right),$$

which is, in fact, the potential of the circle $x^2+y^2=g^2$ on the axial point $(0, 0, c)$; and, observing that the value is independent of f, we have at once the theorem that, considering f as variable, and taking the attracted point at the constant altitude c in the focal hyperbola $\frac{x^2}{f^2-g^2}-\frac{z^2}{g^2}=1$, the potential is the same, whatever is the value of the semi-axis major f of the ellipse.

11. A point in the focal hyperbola determines, with the ellipse, a right circular cone having for its axis the tangent to the hyperbola; viz., the tangent in question is equally inclined to the two lines joining the point with the foci of the hyperbola, or extremities of the major axis of the ellipse. Taking θ for the inclination of the tangent to either of these lines, viz., θ is the semi-aperture of the cone, and γ for the inclination of the tangent to the axis of z, then it is easy to show that

$$\sqrt{c^2+g^2}=c\frac{\cos\gamma}{\sqrt{\cos^2\gamma-\sin^2\theta}};$$

and we thence have

$$V=2\pi c=\left(\frac{\cos\gamma}{\sqrt{\cos^2\gamma-\sin^2\theta}}-1\right),$$

viz., the ellipse is here considered as the section of a right cone of semi-aperture θ, the perpendicular distance from the vertex being $=c$, and the inclination of this distance to the axis of the cone being $=\gamma$; and this being so, the potential is then expressed by the last preceding equation. It will be observed that, $\gamma=\frac{\pi}{2}-\theta$, the section becomes a parabola, and the potential is infinite; for any larger value of γ, the section is a hyperbola, and the formula ceases to be applicable.

12. I originally obtained the result by thus considering the ellipse as the section of a right cone. Consider for a moment, in the case of any cone whatever, the plate included between the plane, perpendicular distance from the vertex $=c$, and the consecutive parallel plane, distance $=c+dc$. Let $d\Sigma$ denote an element of the first plane, r its distance from the vertex, and $r+dr$ the distance produced to meet the second plane; also let $d\omega$ denote the subtended solid angle. We have $d\Sigma\,dc=r^2\,dr\,d\omega$, or, since $\frac{dc}{c}=\frac{dr}{r}$, we obtain $d\Sigma=\frac{1}{c}r^3d\omega$, or $\frac{1}{r}d\Sigma=\frac{1}{c}r^2d\omega$; wherefore the potential of the plane section is $V=\frac{1}{c}\int r^2d\omega$, where r denotes the value at a point of the plane section, and the integration extends over the spherical aperture of the cone.

13. Let the position of r be determined by means of its inclination θ to the axis of the cone, and the azimuth ϕ of the plane through r and the axis of the cone; viz., taking the axis of the cone for the axis of z, suppose, as usual, $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$. We have then, as usual, $d\omega = \sin\theta\, d\theta\, d\phi$; and if the equation of the plane be $x\cos\alpha + y\cos\beta + z\cos\gamma = c$, then the value of r is obtained from the equation

$$r\{(\cos\alpha\cos\phi + \cos\beta\sin\phi)\sin\theta + \cos\gamma\cos\theta\} = c;$$

viz., we have for the potential

$$V = c\int \frac{\sin\theta\, d\theta\, d\phi}{\{(\cos\alpha\cos\phi + \cos\beta\sin\phi)\sin\theta + \cos\gamma\cos\theta\}^2},$$

where the integration is extended over the whole spherical aperture of the cone; viz., in the case of a right cone of semi-aperture θ, the limits are from $\theta = 0$ to $\theta = \theta$ and from $\phi = 0$ to $\phi = 2\pi$.

14. Write

$$(\cos\alpha\cos\phi + \cos\beta\sin\phi)\sin\theta + \cos\gamma\cos\theta = M\cos(\theta - N),$$

where M, N are given functions of ϕ; then we have

$$V = c\int \frac{d\phi}{M^2}\int \frac{\sin\theta\, d\theta}{\cos^2(\theta - N)}$$

and the θ integral is

$$\int \frac{[\sin(\theta - N)\cos N + \cos(\theta - N)\sin N]\, d\theta}{\cos^2(\theta - N)},$$

$$= \cos N\sec(\theta - N) + \sin N\log\tan\{\tfrac{1}{4}\pi + \tfrac{1}{2}(\theta - N)\},$$

which between the limits is

$$= \cos N\{\sec(\theta - N) - \sec N\}$$
$$+ \sin N\{\log\tan[\tfrac{1}{4}\pi + \tfrac{1}{2}(\theta - N)] - \log\tan(\tfrac{1}{4}\pi - \tfrac{1}{2}N)\},$$

θ now denoting the semi-aperture of the right cone. And we have

$$V = c\int \frac{d\phi}{M^2}\left\{\cos N\left(\frac{1}{\cos(N - \theta)} - \frac{1}{\cos N}\right)\right.$$
$$\left. + \sin N[\log\tan\{\tfrac{1}{4}\pi + \tfrac{1}{2}(\theta - N)\} - \log\tan(\tfrac{1}{4}\pi - \tfrac{1}{2}N)]\right\}.$$

We may without loss of generality write $\cos\beta = 0$, and therefore $\cos\alpha = \sin\gamma$, γ being now the inclination of the perpendicular on the plane to the axis of the cone. We thus have

$$\cos\gamma\cos\theta + \sin\gamma\cos\phi\sin\theta = M\cos(\theta - N),$$

that is,

$$\cos\gamma = M\cos N,$$
$$\sin\gamma\cos\phi = M\sin N;$$

whence

$$\tan N = \tan\gamma\cos\phi \quad \text{or} \quad N = \tan^{-1}(\tan\gamma\cos\phi),$$
$$M^2 = \cos^2\gamma + \sin^2\gamma\cos^2\phi = 1 - \sin^2\gamma\sin^2\phi,$$

and

$$\frac{\cos N}{\cos(N - \theta)} = \frac{1}{\cos\theta + \sin\theta\tan\gamma\cos\phi}.$$

15. We have, therefore,

$$V = c\int \frac{d\phi}{1-\sin^2\gamma\sin^2\phi}\left(\frac{1}{\cos\theta+\sin\theta\tan\gamma\cos\phi}-1\right)$$

$$+c\int \frac{d\phi\sin\gamma\cos\phi}{(1-\sin^2\gamma\sin^2\varphi)}\left\{\log\tan\left[\tfrac{1}{4}\pi+\tfrac{1}{2}\theta-\tfrac{1}{2}\tan^{-1}(\tan\gamma\cos\varphi)\right]\right.$$
$$\left.-\log\tan\left[\tfrac{1}{4}\pi-\tfrac{1}{2}\tan^{-1}(\tan\gamma\cos\phi)\right]\right\}.$$

But
$$\int \frac{d\phi\cos\phi}{(1-\sin^2\gamma\sin^2\phi)^{\frac{3}{2}}} = \frac{\sin\phi}{(1-\sin^2\gamma\sin^2\varphi)^{\frac{1}{2}}};$$

and hence second line is

$$c\sin\gamma\frac{\sin\phi}{(1-\sin^2\gamma\sin^2\varphi)^{\frac{1}{2}}}\left\{\log\tan\left[\tfrac{1}{4}\pi+\tfrac{1}{2}\theta-\tfrac{1}{2}\tan^{-1}(\tan\gamma\cos\phi)\right]\right.$$
$$\left.-\log\tan\left[\tfrac{1}{4}\pi-\tfrac{1}{2}\tan^{-1}(\tan\gamma\cos\varphi)\right]\right\}$$

$$-c\sin\gamma\int d\phi\frac{\sin\phi}{(1-\sin^2\gamma\sin^2\phi)^{\frac{1}{2}}}\frac{d}{d\phi}\left\{\log\tan\left[\tfrac{1}{4}\pi+\tfrac{1}{2}\theta-\tfrac{1}{2}\tan^{-1}(\tan\gamma\cos\phi)\right]\right.$$
$$\left.-\log\tan\left[\tfrac{1}{4}\pi-\tfrac{1}{2}(\tan^{-1}\tan\gamma\cos\phi)\right]\right\}.$$

But, restoring for a moment N in place of $\tan^{-1}(\tan\gamma\cos\phi)$, we have

$$\frac{d}{d\phi}\log\tan(\tfrac{1}{4}\pi+\tfrac{1}{2}\theta-\mathrm{N}) = -\frac{d\mathrm{N}}{d\phi}\frac{1}{\cos(\mathrm{N}-\theta)} = \frac{\sin\gamma\cos\gamma\sin\phi}{1-\sin^2\gamma\sin^2\phi}\frac{1}{\cos(\mathrm{N}-\theta)},$$

$$\frac{d}{d\phi}\log\tan(\tfrac{1}{4}\pi-\mathrm{N}) = -\frac{d\mathrm{N}}{d\phi}\frac{1}{\cos\mathrm{N}} = \frac{\sin\gamma\cos\gamma\sin\phi}{1-\sin^2\gamma\sin^2\phi}\frac{1}{\cos\mathrm{N}}.$$

And then, in place of $\frac{1}{\cos(\mathrm{N}-\theta)}-\frac{1}{\cos\mathrm{N}}$, writing

$$\frac{1}{\cos\gamma\sqrt{1-\sin^2\gamma\sin^2\phi}}\left(\frac{1}{\cos\theta+\sin\theta\tan\gamma\cos\phi}-1\right),$$

the expression in question becomes

$$c\sin\gamma\frac{\sin\phi}{(1-\sin^2\gamma\sin^2\phi)^{\frac{1}{2}}}\left\{\log\tan\left[\tfrac{1}{4}\pi+\tfrac{1}{2}\theta-\tfrac{1}{2}\tan^{-1}(\tan\gamma\cos\phi)\right]\right.$$
$$\left.-\log\tan\left[\tfrac{1}{4}\pi-\tfrac{1}{2}\tan^{-1}(\tan\gamma\cos\phi)\right]\right\}$$
$$-c\int d\phi\frac{\sin^2\gamma\sin^2\phi}{1-\sin^2\gamma\sin^2\varphi}\left(\frac{1}{\cos\theta+\sin\theta\tan\gamma\cos\phi}-1\right).$$

And we have

$$V = \frac{c\sin\gamma\sin\phi}{(1-\sin^2\gamma\sin^2\phi)^{\frac{1}{2}}}\left\{\log\tan\left[\tfrac{1}{4}\pi+\tfrac{1}{2}\theta-\tfrac{1}{2}\tan^{-1}(\tan\gamma\cos\phi)\right]\right.$$
$$\left.-\log\tan\left[\tfrac{1}{4}\pi-\tfrac{1}{2}\tan^{-1}(\tan\gamma\cos\phi)\right]\right\}$$
$$+c\int d\phi\left(\frac{1}{\cos\theta+\sin\theta\tan\gamma\cos\phi}-1\right).$$

16. The integral is here

$$= \int d\phi \left\{ \frac{\cos\gamma\,(\cos\theta\cos\gamma - \sin\theta\sin\gamma\cos\phi)}{\cos^2\theta\cos^2\gamma - \sin^2\theta\sin^2\gamma\cos^2\varphi} - 1 \right\}$$

$$= \cos^2\gamma\cos\theta \int \frac{d\phi}{\cos^2\theta\cos^2\gamma - \sin^2\theta\sin^2\gamma\cos^2\phi}$$

$$- \cos\gamma\sin\gamma\sin\theta \int \frac{\cos\phi\, d\phi}{\cos^2\theta\cos^2\gamma - \sin^2\theta\sin^2\gamma\cos^2\phi} - \int d\phi$$

$$= \frac{\cos\gamma}{\sqrt{\cos^2\gamma - \sin^2\theta}} \tan^{-1} \frac{\cos\theta\cos\gamma\tan\phi}{\sqrt{\cos^2\gamma - \sin^2\theta}}$$

$$- \frac{\cos\gamma}{\sqrt{\cos^2\gamma - \sin^2\theta}} \tan^{-1} \frac{\sin\theta\sin\gamma\sin\phi}{\sqrt{\cos^2\gamma - \sin^2\theta}} - \phi,$$

as may be immediately verified.

Hence

$$V = -\frac{c\sin\gamma\sin\phi}{\sqrt{1 - \sin^2\gamma\sin^2\phi}} \left\{ \log\tan\left[\tfrac{1}{4}\pi + \tfrac{1}{2}\theta - \tfrac{1}{2}\tan^{-1}(\tan\gamma\cos\phi)\right] - \log\tan\left[\tfrac{1}{4}\pi - \tfrac{1}{2}\tan^{-1}(\tan\gamma\cos\varphi)\right] \right\}$$

$$+ \frac{c\cos\gamma}{\sqrt{\cos^2\theta - \sin^2\gamma}} \tan^{-1} \frac{\cos\theta\cos\gamma\tan\phi}{\sqrt{\cos^2\theta - \sin^2\gamma}}$$

$$- \frac{c\cos\gamma}{\sqrt{\cos^2\theta - \sin^2\gamma}} \tan^{-1} \frac{\sin\theta\sin\gamma\sin\phi}{\sqrt{\cos^2\theta - \sin^2\gamma}}$$

$$- c\phi,$$

which is to be taken between the limits 0 and 2π; or, what is the same thing, the integral may be taken between the limits 0, π, and multiplied by 2. But as ϕ passes from 0 to π, the arc of the form $\tan^{-1}(A\tan\phi)$ passes through the values 0, $\frac{\pi}{2}$, $-\frac{\pi}{2}$, 0, but the other arc of the form $\tan^{-1} B\sin\phi$ through the values 0, $\frac{\pi}{2}$, $\frac{\pi}{2}$, 0; the first arc gives therefore a term π, the second arc a term 0, and the final result is $V = 2c\pi\left(\frac{\cos\gamma}{\sqrt{\cos^2\gamma - \sin^2\theta}} - 1\right)$, which is right.

The Potential of the Circle.

17. In the case of the circle we have $g = f$; the terms containing a^2, b^2 unite throughout into a single term containing $a^2 + b^2$, and there is obviously no loss of generality in assuming $b = 0$, and so reducing this to a^2; viz., we take the axis of x to pass through the projection of the attracted point, the coordinates of this point being therefore $(a, 0, c)$.

We in fact consider the potential

$$V = \int \frac{dx\,dy}{\sqrt{(a-x)^2+y^2+c^2}}$$

over the circle $x^2+y^2=f^2$; or, writing $x = mf\cos\phi$, $y = mf\sin\phi$, we have $dx\,dy = f^2 m\,dm\,d\phi$, and therefore

$$V = f^2 \int \frac{m\,dm\,d\phi}{\sqrt{a^2+c^2+m^2f^2-2maf\cos\phi}},$$

the integral being taken from $m=0$ to $m=1$, and $\phi=0$ to $\phi=2\pi$.

Writing in the general formula $g=f$ and $b=0$, we have

$$V = 2f^2 \int_\theta^\infty \frac{dt\sqrt{1-\frac{a^2}{t+f^2}-\frac{c^2}{t}}}{(t+f^2)\sqrt{t}},$$

where θ denotes the positive root of the equation

$$1-\frac{a}{\theta+f^2}-\frac{c^2}{\theta}=0;$$

or, observing that

$$\begin{aligned}
1-\frac{a^2}{t+f^2}-\frac{c^2}{t} &= a^2\left(\frac{1}{\theta+f^2}-\frac{1}{t+f^2}\right)+c^2\left(\frac{1}{\theta}-\frac{1}{t}\right)\\
&= (t-\theta)\left\{\frac{a^2}{(\theta+f^2)(t+f^2)}+\frac{c^2}{\theta t}\right\}\\
&= \frac{t-\theta}{t.t+f^2}\left\{\left(1-\frac{c^2}{\theta}\right)t+\frac{c^2}{\theta}(t+f^2)\right\}\\
&= \frac{(t-\theta)\left(t+\frac{c^2f^2}{\theta}\right)}{t.t+f^2},
\end{aligned}$$

we have also

$$V = 2f^2 \int_\theta^\infty \frac{\sqrt{\left(t-\theta.t+\frac{c^2f^2}{\theta}\right)}\,dt}{t(t+f^2)\sqrt{t+f^2}}.$$

18. The present particular case gives rise to some interesting investigations. We may, in the first place, complete the process of first integrating directly in regard to m.

Writing

$$V = f\int \frac{[(mf-a\cos\phi)+a\cos\phi]\,dm\,d\phi}{\{(mf-a\cos\phi)^2+a^2\sin^2\phi+c^2\}^{\frac{1}{2}}},$$

the integral in regard to m is

$$= \frac{1}{f}\left\{\sqrt{(mf-a\cos\phi)^2+a^2\sin^2\phi+c^2} + a\cos\phi\log\left\{mf-a\cos\phi+\sqrt{(mf-a\cos\phi)^2+a^2\sin^2\phi+c^2}\right\}\right\}$$

to be taken from $m=0$ to $m=1$; and we thus obtain

$$V=\int d\phi \left\{ \sqrt{a^2+c^2+f^2-2af\cos\phi} - \sqrt{a^2+c^2} \right.$$
$$+a\cos\phi\,[\log(f-a\cos\phi+\sqrt{a^2+c^2+f^2-2af\cos\phi})$$
$$\left. -\log(-a\cos\phi+\sqrt{a^2+c^2})] \right\}.$$

Writing for shortness $\sqrt{a^2+c^2+f^2-2af\cos\phi}=\Delta$, the second line of this is

$$a\sin\phi\,[\log(f-a\cos\phi+\Delta)-\log(-a\cos\phi+\sqrt{a^2+c^2})]$$
$$-\int d\phi\, a^2\sin^2\phi \left\{ \frac{f+\Delta}{\Delta(f-a\cos\phi+\Delta)} - \frac{1}{-a\cos\phi+\sqrt{a^2+c^2}} \right\},$$

and we thus have

$$V=a\sin\phi\,\{\log(f-a\cos\phi+\Delta)-\log(-a\cos\phi+\sqrt{a^2+c^2})\}$$
$$+\int d\phi \left(\Delta-\sqrt{a^2+c^2}-\frac{a^2\sin^2\phi\,(f+\Delta)}{\Delta(f-a\cos\phi+\Delta)} + \frac{a^2\sin^2\phi}{-a\cos\phi+\sqrt{a^2+c^2}} \right).$$

19. We have $\dfrac{f+\Delta}{\Delta(f-a\cos\phi+\Delta)} = \dfrac{(f+\Delta)(f-a\cos\phi-\Delta)}{\Delta\{(f-a\cos\phi)^2-\Delta^2\}}$,

the numerator of which is $f^2-\Delta^2-a\cos\phi\,(f+\Delta)$,

$$=f^2+\Delta^2+a\cos\phi\,(f-a\cos\phi-\Delta)-2af\cos\phi+a^2\cos^2\phi,$$
$$=-c^2-a^2\sin^2\phi+a\cos\phi\,(f-a\cos\phi-\Delta),$$

and the denominator is $=-c^2-a^2\sin^2\phi$. The second line of V is thus

$$=\int d\phi \left\{ \Delta-\sqrt{a^2+c^2}-\frac{a^2\sin^2\phi}{\Delta} \right.$$
$$\left. +\frac{a^3\sin^2\phi\cos\phi}{\Delta}\,\frac{f-a\cos\phi-\Delta}{c^2+a^2\sin^2\phi} + \frac{a^2\sin^2\phi\,(\sqrt{a^2+c^2}+a\cos\phi)}{c^2+a^2\sin^2\phi} \right\},$$

which is easily reduced to

$$\int d\phi \left\{ \frac{c^2+f^2-af\cos\phi}{\Delta} - \frac{c^2a\cos\phi\,(f-a\cos\phi)}{(c^2+a^2\sin^2\phi)\,\Delta} - \frac{c^2\sqrt{a^2+c^2}}{c^2+a^2\sin^2\phi} \right\},$$

the last term of which is $=-c\tan^{-1}\dfrac{\sqrt{a^2+c^2}\tan\phi}{c}$; and we thus have

$$V=a\sin\phi\,\{\log(f-a\cos\phi+\Delta)-\log(-a\cos\phi+\sqrt{a^2+c^2})\}$$
$$-c\tan^{-1}\frac{\sqrt{a^2+c^2}\tan\phi}{c}$$
$$+\int d\phi \left\{ \frac{c^2+f^2-af\cos\phi}{\Delta} - \frac{c^2a\cos\phi\,(f-a\cos\phi)}{(c^2+a^2\sin^2\phi)\,\Delta} \right\}$$

between the limits 0, 2π; or, finally,

$$V = -2c\pi + 2\int_0^\pi d\phi \left\{ \frac{c^2+f^2-af\cos\phi}{\Delta} - \frac{c^2 a\cos\phi\,(f-a\cos\phi)}{(c^2+a^2\sin^2\phi)\,\Delta} \right\};$$

in partial verification whereof observe that for $a=0$ we have $\Delta = \sqrt{c^2+f^2}$, and the value becomes

$$V = 2\pi\,(\sqrt{c^2+f^2}-c),$$

which, writing therein g in place of f, agrees with a foregoing result.

20. The process applied to finding the Potential of the Ellipse is really applicable step by step to the Circle; but if we begin by assuming $g=f$, it presents itself under a different and simplified form. Starting from

$$V = f^2 \int m\,dm \int \frac{d\phi}{\sqrt{a^2+c^2+m^2f^2-2maf\cos\phi}},$$

for convenience we assume

$$P^2+Q^2 = a^2+c^2+m^2f^2,$$
$$PQ = maf,$$

thereby converting the radical into $\sqrt{P^2+Q^2-2PQ\cos\phi}$. Writing also $\Omega = a^4+c^4+m^4f^4+2a^2c^2+2m^2c^2f^2-2m^2a^2f^2$, $= (P^2-Q^2)^2$, and hence assuming $P^2-Q^2 = \sqrt{\Omega}$, and combining with the foregoing equation $P^2+Q^2 = a^2+c^2+m^2f^2$, we have

$$P^2 = \tfrac{1}{2}(a^2+c^2+m^2f^2+\sqrt{\Omega}),$$
$$Q^2 = \tfrac{1}{2}(a^2+c^2+m^2f^2-\sqrt{\Omega}).$$

21. This being so, the transformation-equations to the new variable T are

$$\cos\phi = \frac{P\cos T+Q}{P+Q\cos T}, \text{ whence } \cos T = \frac{P\cos\phi-Q}{P-Q\cos\phi},$$

$$\sin\phi = \frac{\sqrt[4]{\Omega}\sin T}{P+Q\cos T}, \qquad \sin T = \frac{\sqrt[4]{\Omega}\sin\phi}{P-Q\cos\phi};$$

and also $\sqrt{\Omega} = (P+Q\cos T)\,(P-Q\cos\phi), = P^2-Q^2$.

We find moreover

$$d\phi = \frac{\sqrt[4]{\Omega}\,dT}{P+Q\cos T}, \qquad dT = \frac{\sqrt[4]{\Omega}\,d\phi}{P-Q\cos\phi},$$

and

$$P^2+Q^2-2PQ\cos\phi = \frac{\sqrt{\Omega}\,(P-Q\cos T)}{P+Q\cos T},$$

whence

$$\frac{d\phi}{\sqrt{P^2+Q^2-2PQ\cos\phi}} = \frac{dT}{\sqrt{P^2-Q^2\cos^2 T}},$$

and hence

$$V = f^2 \int m\,dm \int \frac{dT}{\sqrt{P^2-Q^2\cos^2 T}},$$

where the limits of T are from 0 to 2π, or, what is the same thing, we may multiply by 4, and take them to be 0, $\frac{1}{2}\pi$.

22. Assuming next $t = P^2 - m^2f^2 + (P^2 - Q^2)\cot^2 T$,

we have
$$t - P^2 + m^2f^2 = (P^2 - Q^2)\frac{\cos^2 T}{\sin^2 T},$$
$$t - Q^2 + m^2f^2 = (P^2 - Q^2)\frac{1}{\sin^2 T},$$
$$t \quad + m^2f^2 = (P^2 - Q^2\cos^2 T)\frac{1}{\sin^2 T};$$

and thence
$$\sqrt{t - P^2 + m^2f^2 . t - Q^2 + m^2f^2 . t + m^2f^2} = (P^2 - Q^2)\frac{\cos T}{\sin^3 T}\sqrt{P^2 - Q^2\cos^2 T},$$

also
$$dt = -2(P^2 - Q^2)\frac{\cos T}{\sin^3 T}dT;$$

and consequently
$$\frac{dt}{\sqrt{t - P^2 + m^2f^2 . t - Q^2 + m^2f^2 . t + m^2f^2}} = \frac{-2dT}{\sqrt{P^2 - Q^2\cos^2 T}}.$$

$T = 0$ gives $t = \infty$, and $T = \frac{1}{2}\pi$ gives $t = P^2 - m^2f^2, = G$ suppose; and we thus have
$$V = 2f^2 \int m\,dm \int_G^\infty \frac{dt}{\sqrt{t - P^2 + m^2f^2 . t - Q^2 + m^2f^2 . t + m^2f^2}}.$$

23. We have $(t - P^2 + m^2f^2)(t - Q^2 + m^2f^2)$
$$= t^2 + (m^2f^2 - a^2 - c^2)t - m^2c^2f^2,$$

or, putting m^2t in the place of t, this is
$$= m^2\{m^2t^2 + (m^2f^2 - a^2 - c^2)t - c^2f^2\},$$

or, what is the same thing,
$$= m^2t\,(t + f^2)\left\{m^2 - \frac{a^2}{t + f^2} - \frac{c^2}{t}\right\};$$

whence, completing the substitution, we have
$$V = 2f^2 \int m\,dm \int_\theta^\infty \frac{1}{\sqrt{m^2 - \frac{a^2}{t + f^2} - \frac{c^2}{t}}} \quad \frac{dt}{\sqrt{t}\,(t + f^2)},$$

where the inferior limit $\theta, = \frac{1}{m^2}G, = \frac{1}{m^2}P^2 - f^2$ is, in fact, the positive root of the equation
$$m^2 - \frac{a^2}{\theta + f^2} - \frac{c^2}{\theta} = 0.$$

24. We may hence integrate in regard to m, through the sign $\int dt$, in the same way as if θ were constant; viz., we have
$$V = 2f^2\left[\int_\theta^\infty \sqrt{m^2 - \frac{a^2}{t + f^2} - \frac{c^2}{t}} \quad \frac{dt}{\sqrt{t}\,(t + f^2)}\right],$$

where the function of m is to be taken between the limits $0, 1$: for $m = 0$ we have $\theta = \infty$, and the function vanishes; hence, writing $m = 1$, we obtain $$V = 2f^2 \int_\theta^\infty \sqrt{1 - \frac{a^2}{t+f^2} - \frac{c^2}{t}} \; \frac{dt}{\sqrt{t}\,(t+f^2)},$$

where θ now denotes the positive root of $$1 - \frac{a^2}{\theta+f^2} - \frac{c^2}{\theta} = 0.$$

25. But it is interesting to reverse the transformation, so as to bring the radical back into its original form. For this purpose, taking now

$$P^2 + Q^2 = a^2 + c^2 + f^2,$$
$$PQ = af,$$

and consequently
$$P^2 = \tfrac{1}{2}(a^2 + c^2 + f^2 + \sqrt{\Omega}),$$
$$Q^2 = \tfrac{1}{2}(a^2 + c^2 + f^2 - \sqrt{\Omega}),$$

where $$\Omega = a^4 + c^4 + f^4 + 2a^2c^2 + 2c^2f^2 - 2a^2f^2,$$

and writing $$t = P^2 - f^2 + (P^2 - Q^2)\cot^2 T,$$

we first obtain

$$V = f^2 \int_0^{2\pi} \frac{\Omega \cos^2 T \, dT}{(P^2 - Q^2\cos^2 T - f^2 \sin^2 T)\,(P^2 - Q^2\cos^2 T)^{\frac{3}{2}}};$$

and then, writing
$$\cos T = \frac{P\cos\phi - Q}{P - Q\cos\phi},$$
$$\sin T = \frac{\sqrt[4]{\Omega}\sin\phi}{P - Q\cos\phi},$$

we bring in the variable ϕ. But it is important to remark that this is not the quantity which was, at the beginning of the investigation, represented by this letter, and that it is not easy to see the connexion between the two quantities ϕ. We find

$$V = f^2 \int_0^{2\pi} \frac{(P - Q\cos\phi)^2\,(P\cos\phi - Q)^2 d\phi}{(a^2 + c^2 + f^2\cos^2\phi - 2af\cos\phi)\,(a^2 + c^2 + f^2 - 2af\cos\phi)^{\frac{3}{2}}}.$$

26. To reduce this, write as before

$$\Delta = \sqrt{a^2 + c^2 + f^2 - 2af\cos\phi},$$

and also
$$\Phi = a^2 + c^2 - 2af\cos\phi + f^2\cos^2\phi,$$

so that the denominator in the integral is $= \Phi\Delta^3$.
We have

$$\begin{aligned}(P - Q\cos\phi)^2(P\cos\phi - Q)^2 &= (\Delta^2 - Q^2\sin^2\phi)\,(\Delta^2 - P^2\sin^2\phi),\\ &= \Delta^4 - (a^2 + c^2 + f^2)\,\Delta^2\sin^2\phi + a^2f^2\sin^4\phi,\\ &= \Delta^2\{\Delta^2 - (c^2 + f^2)\sin^2\phi\} - a^2\sin^2\phi\,(\Delta^2 - f^2\sin^2\phi),\\ &= \Delta^2\{\Delta^2 - (c^2 + f^2)\sin^2\phi\} - a^2\sin^2\phi\,.\,\Phi,\end{aligned}$$

and hence $$V = \int \frac{f^2[\Delta^2 - (c^2 + f^2)\sin^2\phi]\,d\phi}{\Phi\Delta} - a^2f^2 \int \frac{\sin^2\phi\, d\phi}{\Delta^3},$$

the limits being always $0, 2\pi$. But we have identically

$$\frac{d}{d\phi}\frac{\sin\phi}{\Delta} = \frac{\cos\phi}{\Delta} - \frac{af\sin^2\phi}{\Delta^3},$$

and thence $$\int \frac{\sin^2\phi\, d\phi}{\Delta^3} = -\frac{1}{af}\left(\frac{\sin\phi}{\Delta}\right) - \frac{1}{af}\int \frac{\cos\phi\, d\phi}{\Delta},$$

where the term $\left(\frac{\sin\phi}{\Delta}\right)$ is to be taken between the limits, but for the present I retain it as it stands. Moreover, $\Delta^2 = \Phi + f^2\sin^2\phi$, and consequently $\Delta^2 - (c^2+f^2)\sin^2\phi = \Phi - c^2\sin^2\phi$, and we thus obtain the result

$$V = af\left(\frac{\sin\phi}{\Delta}\right) - af\int\frac{\cos\phi\, d\phi}{\Delta} + f^2\int\frac{d\phi}{\Delta} - c^2f^2\int\frac{\sin^2\phi\, d\phi}{\Phi\Delta},$$

where the denominators under the integral signs are

$$\Delta, = \sqrt{a^2+c^2+f^2-2af\cos\phi}, \text{ and } \Phi\Delta, = (a^2+c^2-2af\cos\phi+f^2\cos^2\phi)\Delta.$$

27. We may, by a transformation such as that for the change of parameter in an elliptic integral of the third kind, make the denominators to be Δ and $(c^2+a^2\sin^2\phi)\Delta$; viz., for this purpose we assume $\Lambda = \tan^{-1}\frac{B\Delta}{A}$, where B and A are functions of ϕ such that we have identically $A^2+B^2\Delta^2 = (c^2+a^2\sin^2\phi)(a^2+c^2-2af\cos\phi+f^2\cos^2\phi)$; the values of B, A are found to be $c\cos\phi$ and $\sin\phi(a^2+c^2-af\cos\phi)$, whence, dividing each of these for greater convenience by $\sin\phi$, we have

$$\Lambda = \tan^{-1}\left(\frac{c\cot\phi\,\Delta}{a^2+c^2-af\cos\phi}\right),$$

so that, writing now $B, A = c\cot\phi$ and $a^2+c^2-af\cos\phi$ respectively, the value is

$$\Lambda = \tan^{-1}\left(\frac{B\Delta}{A}\right),$$

where

$$A^2+B^2\Delta^2 = \frac{1}{\sin^2\phi}\Pi\Phi;$$

and, as before, $\Phi = a^2+c^2-2af\cos\phi+f^2\cos^2\phi$, and also $\Pi = c^2+a^2\sin^2\phi$.

We have $$\frac{d\Lambda}{d\phi} = \frac{(AB'-A'B)\Delta^2 + \frac{1}{2}AB(\Delta^2)'}{(A^2+B^2\Delta^2)\Delta} \quad \left(A' = \frac{dA}{d\phi}, \text{ \&c.}\right);$$

and then

$$AB'-A'B = \frac{a}{\sin^2\phi}(-a^2-c^2+af\cos^3\phi),$$

$$\tfrac{1}{2}AB(\Delta^2)' = \frac{c\sin\phi\cos\phi}{\sin^2\phi}(a^2+c^2-af\cos\phi)\,af\sin\phi,$$

and the numerator thus is

$$\frac{c}{\sin^2\phi}\{(-a^2-c^2+af\cos^3\phi)(a^2+c^2+f^2-2af\cos\phi) + af\cos\phi(1-\cos^2\phi)(a^2+c^2-af\cos\phi)\},$$

which is in fact

$$= \frac{c}{\sin^2\phi}\{-(c^2+a^2\sin^2\phi)(a^2+c^2+f^2-2af\cos\phi) + (af\cos\phi-a^2\cos^2\phi)(a^2+c^2-2af\cos\phi+f^2\cos^2\phi)\},$$

$$= \frac{c}{\sin^2\phi}\{-\Pi\Delta^2+(af\cos\phi-a^2\cos^2\phi)\Phi\};$$

or, what is the same thing,

$$= \frac{c}{\sin^2\phi}\{-\Pi\phi-\Pi f^2\sin^2\phi+(af\cos\phi-a^2\cos^2\phi)\Phi\},$$

and the denominator, by what precedes, is

$$= \frac{1}{\sin^2\phi}\,.\,\Pi\Phi\Delta.$$

We thus have

$$\frac{1}{c}\frac{d\Lambda}{d\phi} = -\frac{1}{\Delta} - \frac{f^2\sin^2\phi}{\Phi\Delta} + \frac{af\cos\phi-a^2\cos^2\phi}{\Pi\Delta},$$

whence, by integration,

$$\frac{1}{c}\tan^{-1}\left(\frac{c\cot\phi\,\Delta}{a^2+c^2-af\cos\phi}\right) = -\int\frac{d\phi}{\Delta} + \int\frac{(af\cos\phi-a^2\cos^2\phi)\,d\phi}{\Pi\Delta} - f^2\int\frac{\sin^2\phi\,d\phi}{\Phi\Delta},$$

which is the required formula of transformation.

28. Multiplying by c^2, and subtracting from the value of V, we find

$$V = c\tan^{-1}\left(\frac{c\cot\phi\,\Delta}{a^2+c^2-af\cos\phi}\right) + af\left(\frac{\sin\phi}{\Delta}\right) + \int\frac{(c^2+f^2-af\cos\phi)\,d\phi}{\Delta} - c^2a\int\frac{\cos\phi\,(f-a\cos\phi)\,d\phi}{(c^2+a^2\sin^2\phi)\,\Delta},$$

which is to be taken between the limits 0, 2π; viz., we thus have

$$V = -2c\pi + 2\int_0^\pi\frac{(c^2+f^2-af\cos\phi)\,d\phi}{\Delta} - 2c^2a\int_0^\pi\frac{\cos\phi\,(f-a\cos\phi)\,d\phi}{(c^2+a^2\sin^2\phi)\,\Delta},$$

agreeing with a former result.

29. But this former result, previous to the final step of taking the integrals between the limits, was

$$V = 2a\sin\phi\log\left(\frac{f-a\cos\phi+\Delta}{-a\cos\phi+\sqrt{a^2+c^2}}\right) - c\tan^{-1}\left(\frac{\sqrt{a^2+c^2}\tan\phi}{c}\right) + \int\frac{(c^2+f^2-af\cos\phi)\,d\phi}{\Delta} - c^2a\int\frac{\cos\phi\,(f-a\cos\phi)\,d\phi}{(c^2+a^2\sin^2\phi)\,\Delta};$$

viz., the integrals are the same, but the integrated terms are altogether different; the explanation of course is, the ϕ's are different in the two

formulæ, which therefore do not correspond element by element, but only in their ultimate value between the limits.

30. In order to discuss numerically the Potential of the Circle,

$$V = 2f^2 \int_\theta^\infty \frac{\sqrt{\left(t-\theta \,.\, t+\frac{c^2f^2}{\theta}\right)}\, dt}{t\,(t+f^2)\,\sqrt{t+f^2}},$$

this must be reduced to elliptic functions. Writing $t = \theta + x^2$, we have

$$V = 4f^2 \int_0^\infty \frac{x^2\sqrt{x^2+\beta^2}\, dx}{(x^2+\theta)\,(x^2+\alpha^2)^{\frac{3}{2}}};$$

if for shortness

$$\theta + f^2 = \alpha^2,$$

$$\theta + \frac{c^2f^2}{\theta} = \beta^2.$$

The constants α, β, θ may be considered as replacing the original constants a, c, f; viz., from the last two equations and the equation

$$\frac{\alpha^2}{\theta+f^2} + \frac{c^2}{\theta} = 1,$$

we deduce $\quad \alpha^2 = \dfrac{\alpha^2\,(\alpha^2-\beta^2)}{\alpha^2-\theta}, \quad c^2 = \dfrac{\theta\,(\beta^2-\theta)}{\alpha^2-\theta}, \quad f^2 = \alpha^2-\theta\,;$

showing that α^2, β^2, θ are in order of decreasing magnitude; viz., $\alpha^2-\beta^2$, $\beta^2-\theta$, $\alpha^2-\beta^2$ are all positive. The formula may be written

$$\tfrac{1}{4}V = (\alpha^2-\theta)\int_0^\infty \frac{x^2\,(x^2+\beta^2)\, dx}{(x^2+\theta)\,(x^2+\alpha^2)\,\sqrt{x^2+\alpha^2\,.\,x^2+\beta^2}};$$

which, in virtue of the identity

$$(\alpha^2-\theta)\,x^2\,(x^2+\beta^2) = (\alpha^2-\theta)\,(x^2+\theta)\,(x^2+\alpha^2)$$
$$-\alpha^2\,(\alpha^2-\beta^2)\,(x^2+\theta) - \theta\,(\beta^2-\theta)\,(x^2+\alpha^2),$$

becomes

$$\tfrac{1}{4}V = (\alpha^2-\theta)\int_0^\infty \frac{dx}{\sqrt{x^2+\alpha^2\,.\,x^2+\beta^2}}$$
$$-\alpha^2\,(\alpha^2-\beta^2)\int_0^\infty \frac{dx}{(x^2+\alpha^2)\,\sqrt{x^2+\alpha^2\,.\,x^2+\beta^2}}$$
$$-\theta\,(\beta^2-\theta)\int_0^\infty \frac{dx}{(x^2+\theta)\,\sqrt{x^2+\alpha^2\,.\,x^2+\beta^2}}.$$

31. Writing here $x = \alpha \cot u$, and therefore $dx = -\alpha \operatorname{cosec}^2 u\, du$, to the values $x = \infty$, 0 correspond $u = 0$, $\frac{1}{2}\pi$, and we have

$$\tfrac{1}{4}V = \int_0^{\frac{1}{2}\pi} \frac{du}{\sqrt{a^2\cos^2 u + \beta^2 \sin^2 u}} \left\{ a^2 - \theta - (a^2 - \beta^2)\sin^2 u - \frac{\theta(\beta^2-\theta)\sin^2 u}{a^2\cos^2 u + \theta \sin^2 u} \right\}$$

$$= \int_0^{\frac{1}{2}\pi} \frac{du}{\sqrt{a^2\cos^2 u + \beta^2 \sin^2 u}} \left\{ a^2 - \theta + \frac{\theta(\beta^2-\theta)}{a^2-\theta} - (a^2-\beta^2)\sin^2 u - \frac{a^2\theta(\beta^2-\theta)}{a^2-\theta} \frac{1}{a^2\cos^2 u + \theta \sin^2 u} \right\}.$$

Writing $k^2 = 1 - \dfrac{\beta^2}{a^2}$, we have

$$\sqrt{a^2\cos^2 u + \beta^2 \sin^2 u} = a\sqrt{1 - k^2 \sin^2 u},$$

and thence

$$\tfrac{1}{4}V = \int_0^{\frac{1}{2}\pi} \frac{du}{\sqrt{1-k^2\sin^2 u}} \times \left\{ a(1-k^2\sin^2 u) - \frac{k^2\theta}{1-\frac{\theta}{a^2}} - \frac{\frac{\theta}{a}\frac{\beta^2-\theta}{a^2}}{1-\frac{\theta}{a^2}} \frac{1}{1-\left(1-\frac{\theta}{a^2}\right)\sin^2 u} \right\};$$

viz., writing $n = -1 + \dfrac{\theta}{a^2}$ (n is negative, and in absolute magnitude < 1), and moreover $\beta^2 = a^2 k'^2$ and $\theta = (n+1)a^2$, this is

$$\tfrac{1}{4}V = \int_0^{\frac{1}{2}\pi} \frac{du}{\sqrt{1-k^2\sin^2 u}} \times \left\{ (1-k^2\sin^2 u)\, a + \frac{k^2}{n}(n+1)\, a - \frac{n+1\,.\,n+k^2}{n}\, a \frac{1}{1+n\sin^2 u} \right\};$$

viz., this is

$$= a\left\{ \mathrm{E}_1 k + k^2 \frac{n+1}{n} \mathrm{F}_1 k - \frac{n+1\,.\,n+k^2}{n} \Pi_1(n,k) \right\}.$$

32. This may be further reduced by substituting for the complete function $\Pi_1(n,k)$, its value; viz., writing

$$n = \left(-1 + \frac{\theta}{a^2}\right) = -1 + k'^2 \sin^2\lambda;$$

that is, $\sin^2\lambda = \dfrac{\theta}{\beta^2}$; then, writing the value first in the form

$$a\left\{ \mathrm{E}_1 k - (n+1)\mathrm{F}_1 k - \frac{n+1\,.\,n+k^2}{n}\left[\Pi_1(n,k) - \mathrm{F}_1 k\right] \right\},$$

and observing that

$$\frac{n+1\,.\,n+k^2}{n}\left[\Pi_1(n,k) - \mathrm{F}_1 k\right] = \frac{k'^4 \sin^2\lambda\cos^2\lambda}{1-k'^2\sin^2\lambda}\left[\Pi_1(n,k) - \mathrm{F}_1 k\right]$$

$$= \frac{k'^2 \sin\lambda\cos\lambda}{\sqrt{1-k'^2\sin^2\lambda}} \left\{ \tfrac{1}{2}\pi + (\mathrm{F}_1 k - \mathrm{E}_1 k)\,\mathrm{F}(k',\lambda) - \mathrm{F}_1 k\,.\,\mathrm{E}(k',\lambda) \right\},$$

this is

$$\tfrac{1}{4}V = a\left\{ E_1 k - k'^2 \sin^2\lambda\, F_1 k - \frac{k'^2 \sin\lambda\cos\lambda}{\sqrt{1-k'^2\sin^2\lambda}} \right.$$
$$\left. \left[\tfrac{1}{2}\pi + (F_1 k - E_1 k)F(k',\lambda) - F_1 k . E(k',\lambda)\right]\right\},$$

where $a^2 = \theta + f^2$, $k^2 = 1 - \dfrac{\beta^2}{a^2}$, $= 1 - \dfrac{\theta + \dfrac{c^2 f^2}{\theta}}{\theta + f^2}$, $= \dfrac{f^2\left(1 - \dfrac{c^2}{\theta}\right)}{\theta + f^2}$;

or, what is the same thing,

$$k = \frac{af}{f^2+\theta}, \quad \sin^2\lambda = \frac{\theta}{\beta^2}, \quad = \frac{1}{1 + \dfrac{c^2 f^2}{\theta^2}},$$

θ being, it will be recollected, the positive root of

$$\frac{a^2}{f^2+\theta} + \frac{c^2}{\theta} = 1.$$

33. Thus in particular $a=0$, we have $\theta = c^2$, and thence

$$a = \sqrt{c^2+f^2}, \quad k=0, \quad k'=1, \quad \sin\lambda = \frac{c}{\sqrt{c^2+f^2}};$$

whence $\tfrac{1}{4}V = \tfrac{1}{2}\pi\sqrt{c^2+f^2}\,\{1 - \sin^2\lambda - \sin\lambda\,(1-\sin\lambda)\}$,

$$= \tfrac{1}{2}\pi\sqrt{c^2+f^2}\,(1-\sin\lambda), \quad = \tfrac{1}{2}\pi\,(\sqrt{c^2+f^2} - c),$$

or $V = 2\pi\,(\sqrt{c+f^2} - c)$, which is right.

34. If $c=0$, a being $>f$, then $\theta = a^2 - f^2$, $k = \dfrac{f}{a}$, $\lambda = \tfrac{1}{2}\pi$, $a = a$; so that, retaining k as standing for its value $\dfrac{f}{a}$, we have

$$\tfrac{1}{4}V = a\,(E_1 k - k'^2 F_1 k), \quad \text{or} \quad V = 4a\,(E_1 k - k'^2 F_1 k),$$

which may easily be verified.

If $c=0$, a being $< f$, then, recurring to the original equation for the determination of θ, viz., $(\theta+f^2)^2\,\theta\left(\dfrac{a^2}{\theta+f^2} + \dfrac{c^2}{\theta} - 1\right) = 0$, which for $c=0$ becomes $\theta\,(\theta+f^2)(\theta - a^2 + f^2) = 0$, we have (as the positive root of this equation) $\theta = 0$; whence $a = f$; also, observing that $1 - \dfrac{c^2}{\theta} = \dfrac{a^2}{f^2}$, $k = \dfrac{a}{f}$, and $\sin^2\lambda\left(= \dfrac{\theta}{\theta + \dfrac{c^2}{\theta} f^2}, \text{ where } \dfrac{c^2}{\theta} \text{ is finite}\right) = 0$, and retaining k to denote its value $= \dfrac{a}{f}$, we obtain $\tfrac{1}{4}V = f E_1 k$, or $V = 4f E_1 k$.

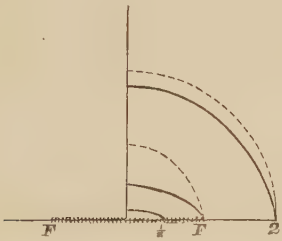
If $a=f$, then in each of the formulæ $k=1$; and since in the first formula $k'^2\mathrm{F}_1k$, k nearly $=1$, is $=k'^2\log\frac{4}{k'}$, vanishing for $k=1$ or $k'=0$, we have $\mathrm{V}=4f\mathrm{E}_1 1$, $=4f$.

It would be interesting to consider the value of the potential at different points of the ellipse $\frac{a^2}{f^2+\theta}+\frac{c^2}{\theta}=1$ (θ constant, a, c current coordinates). For this purpose writing $a=\sqrt{f^2+\theta}\cos q$, $c=\sqrt{\theta}\sin q$, we should have $\alpha=\sqrt{f^2+\theta}$ (a constant), and

$$k=\frac{f\cos q}{\sqrt{f^2+\theta}},\quad k'=\frac{\sqrt{\theta+f^2\sin^2 q}}{\sqrt{f^2+\theta}},$$

$$\sin\lambda=\frac{\sqrt{\theta}}{\sqrt{\theta+f^2\sin^2 q}},\quad \cos\lambda=\frac{f\sin q}{\sqrt{\theta+f^2\sin^2 q}};$$

and then V through k, k', λ, a given function of q.



Section of Equipotential surfaces of a Circle.

35. Suppose, to fix the ideas, $f=1$, and consider the points $(0, c)$ and $(a, 0)$, which have equal potentials. First, if $a>f$ (that is, $a>1$), then writing $k=\frac{1}{a}$, the relation is

$$2\pi\left(\sqrt{1+c^2}-c\right)=\frac{4}{k}\left(\mathrm{E}_1k-k'^2\mathrm{F}_1k\right).$$

$\mathrm{F}_1 30^\circ = 1{\cdot}68575,$
$\mathrm{E}_1 30^\circ = 1{\cdot}46746,$
$\frac{4}{\pi} = 1{\cdot}27324.$

Secondly, if $a<f$ (that is, $a<1$), then writing $k=a$, the relation is

$$2\pi\left(\sqrt{1+c^2}-c\right)=4\mathrm{E}_1k.$$

(1.) In particular $a=\frac{1}{2}$, $=\sin 30^\circ$, this is

$$\sqrt{1+c^2}-c=\frac{2}{\pi}\,\mathrm{E}_1 30^\circ \qquad =\cdot 93421.$$

(2.) $a=1$, then $\sqrt{1+c^2}-c=\frac{2}{\pi}$ $=\cdot 63662.$

(3.) $a=2$, $k=\frac{1}{2}$, $=\sin 30^\circ$,

$$\sqrt{1+c^2}-c=\frac{4}{\pi}\left\{\mathrm{E}_1(30^\circ)-\tfrac{3}{4}\mathrm{F}_1(30^\circ)\right\}=\cdot 25866.$$

But if $\sqrt{1+c^2}-c=m$, then $c=\frac{1}{2}\left(\frac{1}{m}-m\right)$; whence

a	c
0	·0
$\frac{1}{2}$	·06810
1	·46709
2	1·80376

for the values of c, corresponding to the foregoing values of a.

Determination of the Attraction of an Ellipsoidal Shell on an Exterior Point. By Prof. Cayley.

[*Read January 14th*, 1875.]

The shell in question is the indefinitely thin shell included between two concentric, similar, and similarly situated ellipsoidal surfaces, the density being uniform and the attraction varying as the inverse square of the distance.

It was shown by Poisson that the attraction was in the direction of the axis of the circumscribed cone, and expressible in finite terms; the theorem as to the direction of the attraction was afterwards demonstrated geometrically by Steiner, *Crelle*, t. xii. (1834), his method being to divide the shell into elements by means of conical surfaces having their vertices at an interior point Q; and the investigation was about two years ago completed by Prof. Adams, so as to obtain from it the finite expression for the attraction of the shell. The process was explained in a lecture at which I was present: I did not particularly attend to the details of it; and I now reproduce the solution in my own form, stating, in the first place, the geometrical theorems on which it depends.

Statement of the Geometrical Theorems.

1. We consider (see figure, p. 61) an ellipsoid, and two corresponding points, an external point P, and an internal point Q; as will appear, the correspondence is not a reciprocal one. The points are such that each of them is, in regard to the ellipsoid, in the polar plane of the other; moreover PQ is the perpendicular at P to the polar plane of Q; that is, Q being regarded as given, then P is determined as the foot of the perpendicular let fall from Q upon its polar plane; to a given position of Q there corresponds thus a single position of P. It follows that PQ is the normal at P to the confocal ellipsoid through this point; that is,

given the position of P, then Q is the intersection of the polar plane of P by the normal at P to the confocal ellipsoid. Analytically, to a given position of P, there correspond three positions of Q, viz., these are the intersections of the polar plane of P by the normals at P to the three confocal surfaces through this point, and the correspondence of the points P, Q is a (1, 3) correspondence; but the other two positions of Q are external to the ellipsoid, and we are not concerned with them; we determine Q as above by means of the normal to the confocal ellipsoid.

2. If through the point Q we draw at pleasure a chord R′QR″, and join the extremities R′, R″ with P, then the line PQ bisects the angle R′PR″; whence also PR′ : QR′ = PR″ : QR″, or writing QR′, QR″ = r', r'' and PR′, PR″ = ρ', ρ'', then $\frac{\rho'}{r'} = \frac{\rho''}{r''}$. Putting each of these equal ratios $= \frac{\Omega}{R}$, where Ω is a length depending on the position of Q, but independent of the direction of the chord R′QR″, then R will be a length depending on the direction of the chord, and if along the chord (say in the sense Q to R′) we measure off from Q a length QT, = R, thence the locus of the extremity T of this line will be an ellipsoid, centre Q, similarly situate to the given ellipsoid, say this is the "auxiliary ellipsoid."

Consider now the given ellipsoid and a concentric and similarly situated similar ellipsoid, exterior to and indefinitely near it. To fix the ideas, let the semi-axes of the given ellipsoid be mf. mg, mh, and those of the consecutive ellipsoid be $(m+dm)f$, $(m+dm)g$, $(m+dm)h$. Producing the chord R′R″ to meet the consecutive ellipsoid in S′, S″, then the radial thicknesses R′S′, R″S″ of the included shell will be equal to each other, or say each $= \Lambda dm$, where Λ is a quantity dependent as well on the position of the point Q as on the direction of the chord R′R″ through this point.

3. Let 2ϕ denote the angle R′PR″, or, what is the same thing, ϕ denote either of the equal angles R′PQ, R″PQ; then, R, Λ denoting as above, it is found that $\cos\phi = \frac{m\mathrm{R}}{\Lambda}$.

Determination of the Attraction of the Shell.

4. We may now solve the attraction-problem. We consider the indefinitely thin shell (density unity) included between the given ellipsoid and the consecutive ellipsoid, and attracting the exterior point P. We determine the corresponding interior point Q, and then dividing the shell into elements by means of indefinitely thin cones having their vertices at Q, we consider in conjunction the elements

determined by any two opposite cones, say the two opposite cones having for their axis the chord R′QR″, and a spherical aperture $= d\omega$. The shell-element at R′ is

$$r'^2 d\omega \,.\, \mathrm{R'S'} = r'^2 \Lambda d\omega\, dm;$$

its attraction on P is therefore

$$\frac{r'^2}{\rho'^2} \Lambda\, d\omega\, dm, = \frac{\mathrm{R}^2}{\Omega^2} \Lambda\, dm\, d\omega,$$

and the attractions in the directions QR′ and PQ are this quantity multiplied by $\sin\phi$ and $\cos\phi$ respectively.

5. But the shell-element at R″ exerts upon P the same attraction $\frac{\mathrm{R}^2}{\Omega^2} \Lambda\, dm\, d\omega$; and the attractions in the directions QR″ and PQ are this quantity multiplied by $\sin\phi$ and $\cos\phi$ respectively: hence, the attractions in the directions QR′, QR″ exactly counterbalance each other, and there remain only the two equal attractions in the direction PQ; viz., this for either of the elements in question, say for the element at R′, is $= \frac{\mathrm{R}^2}{\Omega^2} \Lambda \cos\phi\; dm\, d\omega,$

or, substituting for $\cos\phi$ its value, $= \frac{m\mathrm{R}}{\Lambda}$, this is

$$= \frac{m\, dm}{\Omega^2} \mathrm{R}^3 d\omega.$$

Hence, the whole attraction of the shell is in the direction PQ, its value being $\frac{m\, dm}{\Omega^2} \iint \mathrm{R}^3 d\omega$

over the whole solid angle at Q; and recollecting that R denotes the radius vector in the auxiliary ellipsoid, we have volume of this ellipsoid $= \iiint r^2 dr\, d\omega = \frac{1}{3} \iint \mathrm{R}^3 d\omega$, that is, $\iint \mathrm{R}^3 d\omega = 3$. volume of auxiliary ellipsoid, $= 4\pi \mathrm{FGH}$, if F, G, H are the semiaxes of the auxiliary ellipsoid. That is,

$$\text{Attraction of shell} = \frac{m\, dm}{\Omega^2} 4\pi \mathrm{FGH}.$$

The problem is now solved; but it remains to prove the geometrical theorems, and to determine the values of the quantities Ω, F, G, H, which enter into the expression for the attraction; and we may also deduce the formula for the attractions of a solid ellipsoid.

Proof of the Geometrical Theorems.

6. I take $\frac{x^2}{f^2} + \frac{y^2}{g^2} + \frac{z^2}{h^2} = m^2$

for the equation of the ellipsoid; a, b, c for the coordinates of P;

ξ, η, ζ for those of Q; α, β, γ for the cosine-inclinations of the radius QR′ to the axes. Hence, in the equation of the ellipsoid, substituting for x, y, z the values $\xi + r\alpha, \eta + r\beta, \zeta + r\gamma$, and writing for shortness

$$A = \frac{\alpha^2}{f^2} + \frac{\beta^2}{g^2} + \frac{\gamma^2}{h^2},$$

$$B = \frac{\alpha\xi}{f^2} + \frac{\beta\eta}{g^2} + \frac{\gamma\zeta}{h^2},$$

$$C = \frac{\xi^2}{f^2} + \frac{\eta^2}{g^2} + \frac{\zeta^2}{h^2} - m^2 \text{ (C being therefore negative)},$$

we have $r', -r''$ as the roots of the equation

$$Ar^2 + 2Br + C = 0;$$

viz.,

$$\frac{2B}{A} = -r' + r'', \quad \frac{C}{A} = -r'r'',$$

and thence

$$r' = \frac{-B + \sqrt{B^2 - AC}}{A}, \quad r'' = \frac{B + \sqrt{B^2 - AC}}{A}, \quad r' + r'' = \frac{2\sqrt{B^2 - AC}}{A}.$$

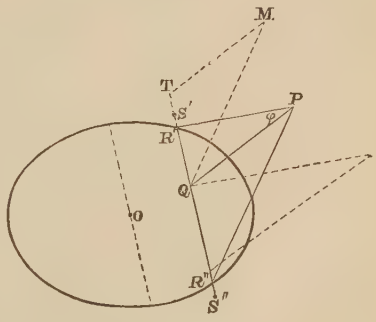
7. Suppose for a moment that the semidiameter parallel to R′R″ is $= mv$, we have evidently $v^2 = \frac{1}{A}$, and then, if in the central section through R′R″ the conjugate semidiameter is mu, the equation of the section referred to these conjugate axes will be $\frac{x^2}{m^2u^2} + \frac{y^2}{m^2v^2} = 1$, or say, $y^2 = m^2v^2 - \frac{v^2}{u^2}x^2$, where y is the coordinate parallel to R′R″, so that, taking the coordinate to belong to the point R′, we have $y = \frac{1}{2}(r' + r'') = \frac{\sqrt{B^2 - AC}}{A}$. For the exterior surface of the shell, m is to be changed into $m + dm$; hence, y, m alone varying, we have

$$y\,dy = mv^2\,dm, = m\,dm\,\frac{1}{A},$$

that is, $dy = m\,dm\,\frac{1}{\sqrt{B^2 - AC}}$;

viz., this is the value of the radial thickness R′S′ of the shell; or, since the same process applies to the point R″, we have

$$R'S' = R''S'' = m\,dm\,\frac{1}{\sqrt{B^2 - AC}},$$



or, calling this, as above, $\Lambda\,dm$, the value of Λ is $= \frac{m}{\sqrt{B^2 - AC}}$.

8. The points P and Q are connected by the condition that, for every direction whatever of the chord R′R″, we have

$$PR' : PR'' = QR' : QR'',$$

or, what is the same thing, that the line QP bisects the angle R′PR″. Taking $PR' = \rho'$, $PR'' = \rho''$, the condition is $\rho' : r' = \rho'' : r''$; and taking (a, b, c) as the coordinates of the point P, we have

$$\rho'^2 = (\xi + r'\alpha - a)^2 + (\eta + r'\beta - b)^2 + (\zeta + r'\gamma - c)^2$$

$$= \sigma^2 + 2r'U + r'^2;$$

if, for shortness, $\sigma^2 = (\xi - a)^2 + (\eta - b)^2 + (\zeta - c)^2 \quad (= \overline{QP}^2)$,

$$U = \alpha(\xi - a) + \beta(\eta - b) + \gamma(\zeta - c),$$

and similarly $\rho''^2 = \sigma^2 - 2r''U + r''^2$.

The required condition therefore is

$$\frac{\sigma^2}{r'^2} + \frac{2}{r'}U = \frac{\sigma^2}{r''^2} - \frac{2}{r''}U,$$

viz., this is $$\sigma^2\left(\frac{1}{r'^2} - \frac{1}{r''^2}\right) + 2U\left(\frac{1}{r'} + \frac{1}{r''}\right) = 0,$$

so that, omitting a factor, it becomes

$$\sigma^2\left(\frac{1}{r'} - \frac{1}{r''}\right) + 2U = 0,$$

that is, $$\sigma^2 \cdot \frac{-2B}{C} + 2U = 0, \text{ or } U = \frac{\sigma^2 B}{C},$$

which must be satisfied independently of the values of α, β, γ.

9. Writing, for greater convenience, $\frac{\sigma^2}{C} = -\theta$, the equation is $U = -\theta B$, viz., substituting for U, B their values, this gives $\xi - a + \frac{\theta\xi}{f^2}$, &c., or say,

$$a = \xi\left(1 + \frac{\theta}{f^2}\right),$$

$$b = \eta\left(1 + \frac{\theta}{g^2}\right),$$

$$c = \zeta\left(1 + \frac{\theta}{g^2}\right),$$

and the assumed relation $\frac{\sigma^2}{C} = -\theta$ is

$$(\xi - a)^2 + (\eta - b)^2 + (\zeta - c)^2 = -\theta\left(\frac{\xi^2}{f^2} + \frac{\eta^2}{g^2} + \frac{\zeta^2}{h^2} - m^2\right);$$

viz., substituting for ξ, η, ζ the foregoing values, and omitting a factor θ, this is

$$\theta\left(\frac{\xi^2}{f^4} + \frac{\eta^2}{g^4} + \frac{\zeta^2}{h^4}\right) = -\left(\frac{\xi^2}{f^2} + \frac{\eta^2}{g^2} + \frac{\zeta^2}{h^2} - m^2\right);$$

or, writing for shortness

$$\frac{1}{\Omega^2} = \left(\frac{\xi^2}{f^4} + \frac{\eta^2}{g^4} + \frac{\zeta^2}{h^4} - m^2\right),$$

the equation is $\theta = -\Omega^2 C.$

We thus see that, (ξ, η, ζ) being given, θ, and therefore also (a, b, c), are uniquely determined. It may be added that, writing $C = -\frac{\sigma^2}{\theta}$, we have $\theta^2 = \Omega^2\sigma^2$, or say $\Omega\sigma = \theta$.

10. We have, moreover, $\frac{a^2}{\theta+f^2} = \frac{\xi^2}{f^4}(\theta+f^2)$, &c.,

and $$\frac{\xi^2}{f^4}(\theta+f^2) + \frac{\eta^2}{g^4}(\theta+g^2) + \frac{\zeta^2}{h^4}(\theta+h^2) = \frac{1}{\Omega^2}\theta + \frac{\xi^2}{f^2} + \frac{\eta^2}{g^2} + \frac{\zeta^2}{h^2};$$

$$= -C + m^2 + C, = m^2;$$

whence $$\frac{a^2}{\theta+f^2} + \frac{b^2}{\theta+g^2} + \frac{c^2}{\theta+h^2} = m^2,$$

or, regarding (a, b, c) as given, θ is determined as a function of (a, b, c) by this cubic equation; and θ being (in accordance with the foregoing equation $\theta = -\Omega^2 C$) assumed to be positive, we have θ the positive root of this equation, and $m^2(\theta+f^2)$, $m^2(\theta+g^2)$, $m^2(\theta+h^2)$ as the squared semiaxes of the confocal ellipsoid through the point P. And θ being known, ξ, η, ζ are, by the foregoing equations $a = \xi\left(1 + \frac{\theta}{f^2}\right)$, &c., determined in terms of ξ, η, ζ; that is, starting from the given external point P, we have the internal point Q. And it appears that PQ is the normal at P to the confocal ellipsoid, or, what is the same thing, the axis of the circumscribed cone, vertex P.

11. The foregoing equation

$$\frac{\xi^2}{f^4}(\theta+f^2) + \frac{\eta^2}{g^4}(\theta+g^2) + \frac{\zeta^2}{h^4}(\theta+h^2) = m^2,$$

considering a, b, c, and therefore θ, as given, shows further that the point Q is situate on an ellipsoid which is the inverse of the confocal ellipsoid $\frac{x^2}{\theta+f^2} + \frac{y^2}{\theta+g^2} + \frac{z^2}{\theta+h^2} = m^2$ in regard to the given ellipsoid $\frac{x^2}{f^2} + \frac{y^2}{g^2} + \frac{z^2}{h^2} = m^2$.

12. Expressing Ω in terms of a, b, c, we have

$$\frac{1}{\Omega^2} = \frac{a^2}{(\theta+f^2)^2} + \frac{b^2}{(\theta+g^2)^2} + \frac{c^2}{(\theta+h^2)^2}.$$

We have $\sigma^2 = \frac{\theta^2}{\Omega^2}$, $= C^2\Omega^2$, and

$$U = \alpha(\xi-a)+\beta(\eta-b)+\gamma(\zeta-c),$$
$$= -\theta\left(\frac{a\xi}{f^2}+\frac{\beta\eta}{g^2}+\frac{\gamma\zeta}{h^2}\right), \ = -\theta B, \ = BC\Omega^2;$$

whence $$\frac{\rho'^2}{r'^2} = \frac{\sigma^2}{r'^2} + 2U\frac{1}{r'} + 1, \ = \frac{C^2\Omega^2}{r'^2} + \frac{2BC\Omega^2}{r'} + 1,$$
$$= C\Omega^2\left(\frac{C}{r'^2}+\frac{2B}{r'}\right)+1;$$

or, since $$A+2B\frac{1}{r'}+C\frac{1}{r'^2}=0,$$

this is $$\frac{\rho'^2}{r'^2} = -AC\Omega^2+1 = \Omega^2\left(\frac{1}{\Omega^2}-AC\right)$$
$$= \frac{\Omega^2}{R^2}, \text{ if } \frac{1}{R^2}=\frac{1}{\Omega^2}-AC;$$

this last equation may also be written

$$\frac{1}{R^2} = \frac{1}{\Omega^2}(a^2+\beta^2+\gamma^2) - C\left(\frac{a^2}{f^2}+\frac{\beta^2}{g^2}+\frac{\gamma^2}{h^2}\right);$$

or, what is the same thing,

$$\frac{1}{R^2} = \frac{a^2}{F^2}+\frac{\beta^2}{G^2}+\frac{\gamma^2}{H^2};$$

if for shortness

$$\frac{1}{F^2} = \frac{1}{\Omega^2}-\frac{C}{f^2},$$
$$\frac{1}{G^2} = \frac{1}{\Omega^2}-\frac{C}{g^2},$$
$$\frac{1}{H^2} = \frac{1}{\Omega^2}-\frac{C}{h^2};$$

viz., substituting herein for C its value $-\frac{\theta}{\Omega^2}$, these equations give

$$F = \frac{\Omega f}{\sqrt{\theta+f^2}}, \quad G = \frac{\Omega g}{\sqrt{\theta+g^2}}, \quad H = \frac{\Omega h}{\sqrt{\theta+h^2}};$$

where Ω stands for its expression in terms of a, b, c.

13. The expression for $\frac{1}{R^2}$ shows that R is the radius vector, cosine-inclinations α, β, γ, in an ellipsoid semi-axes F, G, H, and which may be regarded as having its centre at Q; viz., this is the "auxiliary ellipsoid." And this being so, we have

$$\frac{\rho'}{r'} = \frac{\rho''}{r''} = \frac{\Omega}{R}.$$

It appears from these equations that, drawing from Q parallel to PR″ a line QM, $=\Omega$, and from its extremity M parallel to PQ a line to meet QR′ in T, the locus of T is the auxiliary ellipsoid.

14. By what precedes, the angles R′PQ, R″PQ are equal to each other, say each is $=\phi$; the triangle R′PR″ gives

$$\cos 2\phi = \frac{\rho'^2+\rho''^2-(r'+r'')^2}{2\rho'\rho''},$$

that is,
$$\cos^2\phi = \frac{(\rho'+\rho'')^2-(r'+r'')^2}{4\rho'\rho''};$$

viz., this is
$$= \left(\frac{\Omega^2}{R^2}-1\right)(r'+r'')^2 \div 4\,\frac{\Omega^2}{R^2}\,r'r'',$$

$$= R^2\left(\frac{1}{R^2}-\frac{1}{\Omega^2}\right)\frac{(r'+r'')^2}{4r'r''}$$

$$= -ACR^2\,.\,\frac{4(B^2-AC)}{A^2}\,.\,\frac{-A}{4C}$$

$$= R^2(B^2-AC);$$

or say
$$\cos\phi = R\sqrt{B^2-AC};$$

a remarkable equation which may also be written

$$\cos\phi = \frac{R}{v^2}\,.\,\tfrac{1}{2}(r'+r''),$$

if, as before, v is the semi-diameter parallel to R′R″.

In virtue of the equation $\Lambda = \dfrac{m}{\sqrt{B^2-AC}}$ which defines Λ, the equation becomes
$$\cos\phi = \frac{mR}{\Lambda};$$

and we thus complete the demonstration of the several geometrical theorems upon which the investigation was founded.

Analytical Expressions for the Attraction of the Shell, and for the Resolved Attractions.

15. The attraction of the shell was shown to be

$$= \frac{m\,dm}{\Omega^2}\,4\pi\,FGH;$$

or, since the mass of the shell, density being unity, is

$$\frac{4\pi}{3}\,fgh\,.\,3m^2dm = 4m^2dm\,\pi fgh,$$

we have
$$\text{Attraction} \div \text{Mass} = \frac{1}{m\Omega^2}\,\frac{FGH}{fgh};$$

which, by what precedes, is

$$= \frac{\Omega}{m\sqrt{(f^2+\theta)(g^2+\theta)(h^2+\theta)}},$$

where $$\frac{1}{\Omega^2} = \frac{a^2}{(f^2+\theta)^2} + \frac{b^2}{(g^2+\theta)^2} + \frac{c^2}{(h^2+\theta)^2},$$

θ being the positive root of

$$\frac{a^2}{f^2+\theta} + \frac{b^2}{g^2+\theta} + \frac{c^2}{h^2+\theta} = m^2.$$

16. It is to be observed that the cosine-inclinations of the line PQ to the axes are

$$\frac{a\Omega}{f^2+\theta}, \quad \frac{b\Omega}{g^2+\theta}, \quad \frac{c\Omega}{h^2+\theta}$$

respectively; so that, considering, for instance, the attraction parallel to the axis of x, we have

$$\text{Resolved Attraction} \div \text{Mass} = \frac{a\Omega^2}{m(f^2+\theta)\sqrt{(f^2+\theta)(g^2+\theta)(h^2+\theta)}}.$$

Resolved Attractions of the Ellipsoid $\frac{x^2}{f^2} + \frac{y^2}{g^2} + \frac{z^2}{h^2} = 1.$

17. We may find the attraction of the solid ellipsoid

$$\frac{x^2}{f^2} + \frac{y^2}{g^2} + \frac{z^2}{h^2} = 1.$$

For this purpose, dividing it into shells, semi-axes mf, mg, mh, and $(m+dm)f$, $(m+dm)g$, $(m+dm)h$ respectively; we have for the shell in question

$$\text{Resolved Attraction} \div \text{Mass} = \frac{a\Omega^2}{m(f^2+\theta)\sqrt{(f^2+\theta)(g^2+\theta)(h^2+\theta)}},$$

and the mass of the shell is $\frac{4\pi}{3} fgh \,.\, 3m^2 dm$, where the first factor is the mass of the ellipsoid; whence

$$\text{Resolved Attraction} \div \text{Mass of Ellipsoid} = \frac{a \,.\, 3m\Omega^2 dm}{(f^2+\theta)\sqrt{(f^2+\theta)(g^2+\theta)(h^2+\theta)}},$$

θ being here a function of m, and m extending from 0 to 1. But taking θ as the variable in place of m, the equation

$$\frac{a^2}{f^2+\theta} + \frac{b^2}{g^2+\theta} + \frac{c^2}{h^2+\theta} = m^2$$

gives $$-\frac{1}{\Omega^2} d\theta = 2m dm; \text{ that is, } 3m\Omega^2 dm = -\tfrac{3}{2} d\theta.$$

Moreover $m=0$ gives $\theta=\infty$, and $m=1$ gives $\theta=$ its value as defined by the equation $\dfrac{a^2}{f^2+\theta}+\dfrac{b^2}{g^2+\theta}+\dfrac{c^2}{h^2+\theta}=1$,

so that, reversing the sign, the limits are ∞, θ; or, finally, writing under the integral sign ϕ in place of θ, the formula is

Resolved Attraction ÷ Mass of Ellipsoid

$$=\tfrac{3}{2}a\int_{\theta}^{\infty}\frac{d\phi}{(f^2+\phi)\sqrt{(f^2+\phi)(g^2+\phi)(h^2+\phi)}},$$

which is a known formula.

On the Solution of Linear Differential Equations in Series.

By J. Hammond.

[*Read January 14th*, 1875.]

By Leibnitz's theorem,

$$\mathrm{D}^m\{\phi(x)y\}=\left\{\phi(x)\mathrm{D}^m+m\phi'(x)\mathrm{D}^{m-1}+\frac{m(m-1)}{\lfloor 2}\phi''(x)\mathrm{D}^{m-2}+\ldots\right\}y$$
$$=\phi(x+d)\mathrm{D}^m y,$$

where $d\mathrm{D}^m=m\mathrm{D}^{m-1}$, $d^2\mathrm{D}^m=m(m-1)\mathrm{D}^{m-2}$, ...,

and d operates on D only.

Thus the equation

$$\{\phi_0(x)\mathrm{D}^n+\phi_1(x)\mathrm{D}^{n-1}+\ldots+\phi_n(x)\}y=\phi(x) \quad \ldots\ldots\ldots\ldots (1),$$

when differentiated m times, gives

$$\{[\phi_0(x+d)\mathrm{D}^m]\mathrm{D}^n+[\phi_1(x+d)\mathrm{D}^m]\mathrm{D}^{n-1}+\ldots+\phi_n(x+d)\mathrm{D}^m\}y=\phi^m(x).$$

Now suppose $y=y_0+y_1x+y_2\dfrac{x^2}{\lfloor 2}+\ldots\ldots;$

then $y_0, y_1, \ldots y_{n-1}$ are arbitrary constants, and $y_n, y_{n+1}, \ldots$ are found from the equation

$$\{\mathrm{D}^n\phi_0(d)+\mathrm{D}^{n-1}\phi_1(d)+\ldots+\phi_n(d)\}\mathrm{D}^m y_0=\phi^m(0) \quad \ldots\ldots\ldots (2)$$

by putting $m=0, 1, 2, \ldots,$ and $\mathrm{D}^\kappa y_0=y_\kappa$.

Now $\phi(d)=\phi+d\phi'+\dfrac{d^2}{\lfloor 2}\phi''+\ldots\ldots,$

where $\phi, \phi', \phi''\ldots$ are written instead of $\phi(0), \phi'(0), \phi''(0)\ldots$ for shortness.

Thus (2) becomes

$$m_0y_{m+n}+m_1y_{m+n-1}+\ldots+m_{m+n}y_0=\phi^m \quad \ldots\ldots\ldots\ldots\ldots (3).$$

The general coefficient in (3) is

$$m_\kappa = \phi_\kappa + m\phi'_{\kappa-1} + \frac{m\,(m-1)}{\underline{|2}}\phi''_{\kappa-2} + \ldots\ldots,$$

ϕ_κ, ϕ'_κ, ϕ''_κ ... being all zero for all values of κ not included among $\kappa = 0, 1, 2, \ldots n$.

Thus, if $\quad p\phi_\kappa = \phi'_{\kappa-1}$, $\; p^2\phi_\kappa = \phi''_{\kappa-2}$ ……,

$$m_\kappa = (1+p)^m\phi_\kappa,$$

and (3) becomes

$$\phi_0 y_{m+n} + (1+p)^m(\phi_1 y_{m+n-1} + \phi_2 y_{m+n-2} + \ldots + \phi_{m+n}y_0) = \phi^m \ldots\ldots (4),$$

p operating on ϕ only.

Now write $\quad (m, \kappa)$ for $(1+p)^m\phi_\kappa$;

and A_m for $(1+p)^m(\phi_{m+1}y_{n-1} + \phi_{m+2}y_{n-2} + \ldots\ldots + \phi_{m+n}y_0) - \phi^m$;

therefore, from (4),

$$\mathrm{A}_m + (m, m)\,y_n + (m, m-1)\,y_{n+1} + \ldots\ldots + (m, 1)\,y_{m+n-1} + \phi_0 y_{m+n} = 0.$$

This is true for all positive integral values of m. Thus, putting $m = 0, 1, 2, \ldots\ldots$ in succession,

$$\mathrm{A}_0 + \phi_0 y_n = 0,$$

$$\mathrm{A}_1 + (1, 1)\,y_n + \phi_0 y_{n+1} = 0,$$

$$\mathrm{A}_2 + (2, 2)\,y_n + (2, 1)\,y_{n+1} + \phi_0 y_{n+2} = 0,$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\mathrm{A}_m + (m, m)\,y_n + (m, m-1)\,y_{n+1} + \ldots\ldots + \phi_0 y_{m+n} = 0,$$

solving these equations,

$$y_{m+n} \begin{vmatrix} \mathrm{A}_0, & \phi_0, & 0, & 0, & \ldots\ldots 0 \\ \mathrm{A}_1, & (1,1), & \phi_0, & 0, & \ldots\ldots 0 \\ \mathrm{A}_2, & (2,2), & (2,1), & \phi_0, & \ldots\ldots 0 \\ \mathrm{A}_3, & (3,3), & (3,2), & (3,1), & \ldots\ldots 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \mathrm{A}_m, & (m,m), & (m,m-1), & (m,m-2), & \ldots (m,1) \end{vmatrix} = (-1)^{m+1} \begin{vmatrix} \phi_0, & 0, & 0, & \ldots\ldots 0 \\ (1,1), & \phi_0, & 0, & \ldots\ldots 0 \\ (2,2), & (2,1), & \phi_0, & \ldots\ldots 0 \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ (m,m), & (m,m-1), & (m,m-2), & \ldots \phi_0 \end{vmatrix}$$

$$= \left(-\frac{1}{\phi_0}\right)^{m+1}.$$

Now $\quad \mathrm{A}_0 = \phi_n y_0 + \phi_{n-1}y_1 + \ldots\ldots + \phi_1 y_{n-1} - \phi,$

$$\mathrm{A}_1 = (1, n+1)\,y_0 + (1, n)\,y_1 + \ldots\ldots + (1, 2)\,y_{n-1} - \phi',$$

$$\mathrm{A}_2 = (2, n+2)\,y_0 + (2, n+1)\,y_1 + \ldots\ldots + (2, 3)\,y_{n-1} - \phi'',$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

and $\quad (m, \kappa) = \phi_\kappa + m\phi'_{\kappa-1} + \frac{m\,(m-1)}{\underline{|2}}\phi''_{\kappa-2} + \ldots\ldots$

therefore $\quad (1, n+1) = \phi'_n, \quad (2, n+2) = \phi''_n, \quad \ldots\ldots$

Thus, the expansion of y is

$$y = y_0\left\{1-\frac{1}{\phi_0}\cdot\phi_n\frac{x^n}{\lfloor\underline{n}}+\left(\frac{1}{\phi_0}\right)^2\begin{vmatrix}\phi_n & \phi_0\\ \phi_n'(1,1) & \phi_0\end{vmatrix}\frac{x^{n+1}}{\lfloor\underline{n+1}}-\left(\frac{1}{\phi_0}\right)^3\begin{vmatrix}\phi_n & \phi_0 & 0\\ \phi_n'(1,1) & \phi_0 & \\ \phi_n''(2,2) & (2,1) & \end{vmatrix}\frac{x^{n+2}}{\lfloor\underline{n+2}}+\ldots\right\}$$

$$+y_1\left\{x-\frac{1}{\phi_0}\cdot\phi_{n-1}\frac{x^n}{\lfloor\underline{n}}+\left(\frac{1}{\phi_0}\right)^2\begin{vmatrix}\phi_{n-1} & \phi_0\\ (1,n) & (1,1)\end{vmatrix}\frac{x^{n+1}}{\lfloor\underline{n+1}}-\left(\frac{1}{\phi_0}\right)^3\begin{vmatrix}\phi_{n-1} & \phi_0 & 0\\ (1,n) & (1,1) & \phi_0\\ (2,n+1) & (2,2) & (2,1)\end{vmatrix}\frac{x^{n+2}}{\lfloor\underline{n+2}}+\ldots\right\}$$

$$+\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots$$

$$+y_{n-1}\left\{\frac{x^{n-1}}{\lfloor\underline{n-1}}-\frac{1}{\phi_0}\cdot\phi_1\frac{x^n}{\lfloor\underline{n}}+\left(\frac{1}{\phi_0}\right)^2\begin{vmatrix}\phi_1 & \phi_0\\ (1,2) & (1,1)\end{vmatrix}\frac{x^{n+1}}{\lfloor\underline{n+1}}-\left(\frac{1}{\phi_0}\right)^3\begin{vmatrix}\phi_1 & \phi_0 & 0\\ (1,2) & (1,1) & \phi_0\\ (2,3) & (2,2) & (2,1)\end{vmatrix}\frac{x^{n+2}}{\lfloor\underline{n+2}}+\ldots\right\}$$

$$+\frac{1}{\phi_0}\cdot\phi\frac{x^n}{\lfloor\underline{n}}-\left(\frac{1}{\phi_0}\right)^2\begin{vmatrix}\phi & \phi_0\\ \phi'(1,1) & \end{vmatrix}\frac{x^{n+1}}{\lfloor\underline{n+1}}+\left(\frac{1}{\phi_0}\right)^3\begin{vmatrix}\phi & \phi_0 & 0\\ \phi'(1,1) & \phi_0 & \\ \phi''(2,2) & (2,1) & \end{vmatrix}\frac{x^{n+2}}{\lfloor\underline{n+2}}-\ldots \quad\ldots\ldots (5).$$

When $\phi_0 = 0$, this expansion fails.

In this case, put $x = z+h$, where h is not a root of $\phi_0(h) = 0$.

Then (1) becomes $\phi_0(z+h)\dfrac{d^n y}{dz^n}+\ldots\ldots = \phi(z+h)$,

and y can be expanded in powers of z.

The equation $\phi(x)y = \psi(x)$ ..... (6),

when treated in the same way as (1), gives

$$\frac{\psi(x)}{\phi(x)} = \frac{\psi}{\phi}-\left(\frac{1}{\phi}\right)^2\begin{vmatrix}\psi & \phi\\ \psi' & \phi'\end{vmatrix}x+\left(\frac{1}{\phi}\right)^3\begin{vmatrix}\psi & \phi & 0\\ \psi' & \phi' & \phi\\ \psi'' & \phi'' & 2\phi'\end{vmatrix}\frac{x^2}{\lfloor\underline{2}}-\left(\frac{1}{\phi}\right)^4\begin{vmatrix}\psi & \phi & 0 & 0\\ \psi' & \phi' & \phi & 0\\ \psi'' & \phi'' & 2\phi' & \phi\\ \psi''' & \phi''' & 3\phi'' & 3\phi'\end{vmatrix}\frac{x^3}{\lfloor\underline{3}}+\ldots (7).$$

This may be deduced from (5) by putting $n = 0$,

$$\phi = \psi,\quad \phi' = \psi'\ldots\ldots,$$
$$\phi_0 = \phi,\quad \phi_0' = \phi'\ldots\ldots$$

and $$(m,\kappa) = \frac{\lfloor m}{\lfloor\kappa\ \lfloor m-\kappa}\phi^\kappa.$$

Again, putting $\psi(x) = 1$ in (6) and (7),

$$\frac{1}{\phi(x)} = \frac{1}{\phi}-\left(\frac{1}{\phi}\right)^2\phi'.x+\left(\frac{1}{\phi}\right)^3\begin{vmatrix}\phi' & \phi\\ \phi'' & 2\phi'\end{vmatrix}\frac{x^2}{\lfloor\underline{2}}-\left(\frac{1}{\phi}\right)^4\begin{vmatrix}\phi' & \phi & 0\\ \phi'' & 2\phi' & \phi\\ \phi''' & 3\phi'' & 3\phi'\end{vmatrix}\frac{x^3}{\lfloor\underline{3}}+\ldots (8).$$

The coefficients of (7) suggest that

$$\mathrm{D}^m\left\{\frac{\psi(x)}{\phi(x)}\right\} = \frac{(-1)^m}{\{\phi(x)\}^{m+1}}\begin{vmatrix}\psi(x) & \phi(x) & 0 & \ldots\\ \psi'(x) & \phi'(x) & \phi(x) & \ldots\\ \psi''(x) & \phi''(x) & 2\phi'(x) & \ldots\\ \ldots & \ldots & \ldots & \ldots\\ \ldots & \ldots & \ldots & \ldots\end{vmatrix}\ldots\ldots\ldots\ldots (9),$$

there being $(m+1)$ rows in the determinant, and the last row being

$$\psi^m(x),\ \phi^m(x),\ m\phi^{m-1}(x),\ \frac{m(m-1)}{\lfloor 2}\phi^{m-2}(x),\ \ldots\ldots\ m\phi'(x).$$

Here (9) may be proved in exactly the same way, except that x is not made $=0$ in the course of the work.

There are some particular cases worth notice.

Case I.—Equation (4) reduces to $\phi_0 y_{m+n}+(1+p)^m y_{m+n-r}=0$; therefore $(1+p)^m\phi_\kappa=0$ when κ is not 0 or r.

Putting $m=0, 1, 2, \ldots\ldots$ in succession,

$$\left.\begin{array}{llll} \phi_\kappa & =0 & & \\ \phi'_{\kappa-1} & =-\phi_\kappa & =0 & \\ \phi''_{\kappa-2} & =\phi_\kappa & =0 & \\ \ldots & \ldots & \ldots & \ldots \\ \phi_0^\kappa & =(-1)^\kappa\phi_\kappa & =0 & \end{array}\right\}\ \text{when } \kappa \text{ is not 0 or } r.$$

Thus

$$\phi_0(x) = \phi_0+\frac{x^r}{\lfloor r}\phi_0^r = c\left(1+a_0\frac{x^r}{\lfloor r}\right),$$

$$\phi_1(x) = \frac{x^{r-1}}{\lfloor r-1}\phi_1^{r-1} = ca_1\frac{x^{r-1}}{\lfloor r-1},$$

$$\phi_2(x) = \frac{x^{r-2}}{\lfloor r-2}\phi_2^{r-2} = ca_2\frac{x^{r-2}}{\lfloor r-2},$$

$$\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots$$

$$\phi_r(x) = \text{const.} = ca_r,$$

$$\phi_{r+1}(x)=0,\quad \phi_{r+2}(x)=0\ldots\ldots$$

Equation (1) then becomes, when $r<n$,

$$\left\{\left(1+a_0\frac{x^r}{\lfloor r}\right)\mathrm{D}^n+a_1\frac{x^{r-1}}{\lfloor r-1}\mathrm{D}^{n-1}+\ldots\ldots+a_r\mathrm{D}^{n-r}\right\}y=0\ \ldots\ldots\ (10);$$

and when $r>n$,

$$\left\{\left(1+a_0\frac{x^r}{\lfloor r}\right)\mathrm{D}^n+a_1\frac{x^{r-1}}{\lfloor r-1}\mathrm{D}^{n-1}+\ldots\ldots+a_n\frac{x^{r-n}}{\lfloor r-n}\right\}y=0\ \ldots\ldots\ (11).$$

It is easily seen that the solution of (10) is obtained from that of (11) by solving the equation (11) when n is put $=r$, and then integrating the result $n-r$ times.

Comparing (11) with (1),

$$\phi_0=1,$$

$$\phi_0^r=a_0,\quad \phi_1^{r-1}=a_1,\quad \phi_2^{r-2}=a_2,\ \ldots\ldots$$

Also $(m, \kappa) = 0$ except when $\kappa = r$. And

$$(m, r) = \phi_r + m\phi'_{r-1} + \frac{m(m-1)}{\lfloor 2}\phi''_{r-2} + \&c.$$

$$= a_r + ma_{r-1} + \frac{m(m-1)}{\lfloor 2}a_{r-2} + \ldots\ldots + \frac{m(m-1)\ldots(m-r+1)}{\lfloor r}a_0.$$

And since r is to be taken not less than n, a_r, a_{r-1}, ... a_{n+1} are all zero; therefore

$$(m, r) = \left\{ a_0 \frac{\lfloor m}{\lfloor r \lfloor m-r} + a_1 \frac{\lfloor m}{\lfloor r-1 \lfloor m-r+1} + \ldots + a_n \frac{\lfloor m}{\lfloor r-n \lfloor m-r+n} \right\}.$$

Thus, expanding y by means of the relation

$$y_{m+n} = -(m, r)\, y_{m+n-r},$$

$$y = y_0 \left\{ 1 - \frac{(r-n, r)}{\lfloor r} x^r + \frac{(2r-n, r)(r-n, r)}{\lfloor 2r} x^{2r} - \ldots\ldots \right\}$$

$$+ y_1 \left\{ x - \frac{(r-n+1, r)}{\lfloor r+1} x^{r+1} + \frac{(2r-n+1, r)(r-n+1, r)}{\lfloor 2r+1} x^{2r+1} - \ldots\ldots \right\}$$

$$+ y_2 \left\{ \frac{x^2}{\lfloor 2} - \frac{(r-n+2, r)}{\lfloor r+2} x^{r+2} + \frac{(2r-n+2, r)(r-n+2, r)}{\lfloor 2r+2} x^{2r+2} - \ldots\ldots \right\}$$

$$+ \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$+ y_{n-1} \left\{ \frac{x^{n-1}}{\lfloor n-1} - \frac{(r-1, r)}{\lfloor n+r-1} x^{n+r-1} + \frac{(2r-1, r)(r-1, r)}{\lfloor n+2r-1} x^{n+2r-1} - \ldots\ldots \right\}.$$

Many well known expansions are particular cases of the solution of (10) and (11).

Thus

$\{(1+x)\mathrm{D} - n\}y = 0$	gives the expansion of	$(1+x)^n$,
$\{(1+x)\mathrm{D}^2 + \mathrm{D}\}y = 0$	„ „	$\log(1+x)$,
$\{(1+x^2)\mathrm{D}^2 + 2x\mathrm{D}\}y = 0$	„ „	$\tan^{-1}x$,
$\{(1-x^2)\mathrm{D}^2 - x\mathrm{D}\}y = 0$	„ „	$\sin^{-1}x$,
$\{(1-x^2)\mathrm{D}^2 - x\mathrm{D} + m^2\}y = 0$	„ „	$\mathrm{A}\sin(m\sin^{-1}x) + \mathrm{B}\cos(m\sin^{-1}x)$.

Case II.—The general coefficient of equation (4) is

$$(1+p)^m \phi_\kappa = a_\kappa \mathrm{F}(m+n)\,\mathrm{F}(m+n-1)\ldots\ldots\mathrm{F}(m+n-\kappa+1);$$

therefore, putting $m=0$,

$$\phi_\kappa = a_\kappa \mathrm{F}(n)\,\mathrm{F}(n-1)\ldots\ldots\mathrm{F}(n-\kappa+1).$$

And (4) reduces to

$$a_0 u_{m+n} + a_1 u_{m+n-1} + \ldots\ldots + a_n u_m = 0 \quad \ldots\ldots\ldots\ldots (12),$$

where $$u_{m+n} = \frac{y_{m+n}}{\mathrm{F}(m+n)\,\mathrm{F}(m+n-1)\ldots}.$$

Here $$(1+p)^m = \mathrm{E}_n^m, \text{ and } p = \Delta_n.$$

Now $$\phi_{\kappa-r}^r = p^r \phi_\kappa = \Delta_n^r \phi_\kappa;$$

therefore $$\phi_\kappa^r = \mathrm{E}_\kappa^r \Delta_n^r \phi_\kappa.$$

Thus $$\phi_\kappa(x) = e^{x\mathrm{E}_\kappa \Delta_n} \phi_\kappa \quad \ldots\ldots\ldots\ldots\ldots\ldots (13).$$

When $\mathrm{F}(n)=1$, this is the case of linear differential equations with constant coefficients; and the general coefficient in the solution of

$$(a_0 \mathrm{D}^n + a_1 \mathrm{D}^{n-1} + \ldots\ldots + a_n)\, y = 0$$

is the general solution of the difference equation

$$a_0 y_{m+n} + a_1 y_{m+n-1} + \ldots\ldots + a_n y_m = 0.$$

When $\mathrm{F}(n) = n+b-1$,

$$\phi_\kappa = a_\kappa (n+b-1)(n+b-2)\ldots\ldots(n+b-\kappa) = a_\kappa \frac{\Gamma(n+b)}{\Gamma(n+b-\kappa)},$$

and $$\mathrm{E}_\kappa^r \Delta_n^r \phi_\kappa = a_{\kappa+r}(\kappa+r)(\kappa+r-1)\ldots\ldots(\kappa+1)\frac{\Gamma(n+b)}{\Gamma(n+b-\kappa)};$$

therefore, expanding (13) and putting $\kappa=0$,

$$\phi_0(x) = a_0 + a_1 x + a_2 x^2 + \ldots\ldots + a_n x^n.$$

The general value of $\phi_\kappa(x)$, obtained by expanding (13), is

$$\begin{aligned}
\phi_\kappa(x) &= \frac{\Gamma(n+b)}{\Gamma(n+b-\kappa)}\left\{ a_\kappa + a_{\kappa+1}(\kappa+1)x + a_{\kappa+2}\frac{(\kappa+2)(\kappa+1)}{\lfloor 2}x^2 + \ldots \right\} \\
&= \frac{\Gamma(n+b)}{\Gamma(n+b-\kappa)\lfloor \kappa}\left\{ a_\kappa \lfloor \kappa + a_{\kappa+1}\frac{\lfloor \kappa+1}{}x + a_{\kappa+2}\frac{\lfloor \kappa+2}{\lfloor 2}x^2 + \ldots \right. \\
&\qquad \left. \ldots + a_n \frac{\lfloor n}{\lfloor n-\kappa}x^{n-\kappa} \right\} \\
&= \frac{\Gamma(n+b)}{\Gamma(n+b-\kappa)\lfloor \kappa}\mathrm{D}^\kappa \phi_0(x).
\end{aligned}$$

Thus equation (1) becomes

$$\left\{ \frac{\phi_0(x)}{\Gamma(n+b)}\mathrm{D}^n + \frac{\phi_0'(x)}{\Gamma(n+b-1)}\mathrm{D}^{n-1} + \frac{\phi_0''(x)}{\Gamma(n+b-2)\lfloor 2}\mathrm{D}^{n-2} + \ldots \right.$$
$$\left. \ldots + \frac{\phi_0^n(x)}{\Gamma(b)\lfloor n} \right\} y = 0 \quad \ldots\ldots\ldots\ldots\ldots (14),$$

where $\phi_0(x)$ is any rational integral function of x of the nth degree. And equation (12) becomes

$$\frac{a_0 y_{m+n}}{\Gamma(m+n+b)} + \frac{a_1 y_{m+n-1}}{\Gamma(m+n+b-1)} + \ldots\ldots + \frac{a_n y_m}{\Gamma(m+b)} = 0 \quad \ldots (15).$$

Thus if α be a single root of the equation $\phi_0\left(\frac{1}{x}\right) = 0$, the corresponding solution of (15) is

$$y_m = \Gamma(m+b)\, A\alpha^m,$$

and the corresponding expansion of y from (14)

$$y = A\left\{\Gamma(b) + \Gamma(b+1)\,\alpha x + \Gamma(b+2)\,\frac{\alpha^2 x^2}{\lfloor 2} + \ldots\ldots\right\} \ldots\ldots (16).$$

The n arbitrary constants A, ... are not $y_0, y_1, \ldots$, but are connected with them by linear equations.

When $b=1$, equation (14) reduces to

$$\left\{\varphi_0(x)\,D^n + n\phi_0'(x)\,D^{n-1} + \frac{n(n-1)}{\lfloor 2}\,\phi_0''(x)\,D^{n-2} + \ldots\ldots\right\} y = 0,$$

or

$$D^n\{\phi_0(x)\,y\} = 0.$$

This gives the expansion for rational fractions; and in the same manner, when b is put $=0$, or any positive or negative integer, the expansion obtained is that of $D^{b-1}(f)$, where f stands for the rational fraction, and $D^{b-1}(f)$ for its $(b-1)^{\text{th}}$ differential coefficient; negative indices of course meaning integrations.

The Diagonal Scale Principle applied to Angular Measurement in the Circular Slide Rule. By John R. Campbell.

[*Abstract of Paper, read January 14th*, 1875.]

Before entering upon the construction of diagonal scales, having, in place of the usual equidistant parallel lines crossed by a straight diagonal, as many equidistant concentric arcs of circles crossed by a curved diagonal, it will be necessary for me briefly to describe the instrument, or rather the somewhat rough home-made model of one, in which I have introduced them.

It is simply a form of circular slide rule, combining in one arrangement both the ordinary principle of two logometric scales for multiplication and division, and that introduced by the late Dr. Roget (*vide* Phil. Trans., Nov. 17, 1814) for the finding of powers and roots.

Fig. 1 is a plan of the face; fig. 2, a section of the instrument by a vertical plane through the centre.

The face is a circular cardboard surface AA, 12 or 14 inches in diameter, but which might well be made of smaller dimensions. It consists of two parts,—an annular rim AB for the *fixed* scale, and a circular disc BCB corresponding to the *slide*, turning on an axis C in

Fig. 1.

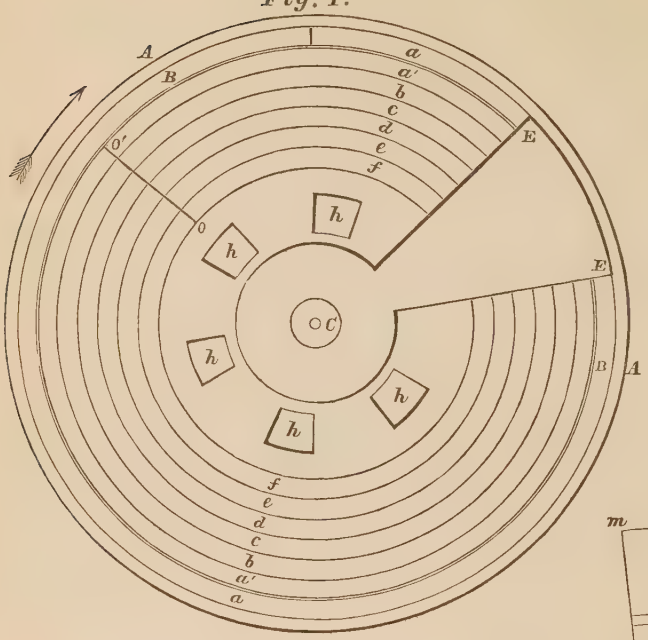


Fig. 2.

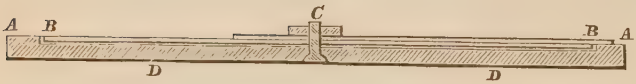


Fig. 4.

Fig. 5.

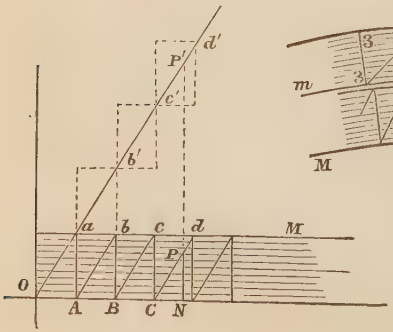
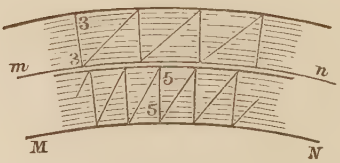


Fig. 3.



the centre of the rim. The rim is mounted on a wooden backing DD, through which the axis passes. For an axis I have used a common pin, but a small bolt, capped by a milled-edged nut, would be better. *aa*, *a′a′*, *bb*, *cc*, *dd*, &c., are scales contained between equidistant circles described from the common centre C. *aa* on the rim is logometrically graduated, or, in other words, a scale of log x. Of those on the disc used in conjunction with it, *a′a′* is also a scale of log x; *bb*, *cc*, and *dd* portions of a scale of log log x. (In Dr. Roget's instrument the scale of log log x is continuous in the form of a spiral—a neater construction than mine, but involving much more labour, and therefore possible inaccuracy in its execution). Other scales, *e.g.*, of log sin θ, log tan θ, may be added in the spaces *ee*, *ff*, &c.; but those already named appear to me of most general importance, and a greater number is liable to produce confusion; to prevent which as much as possible, the rings *aa*, *a′a′*, *bb*, &c., are tinted of various contrasting colours,—red, violet, orange, &c.; or, if left white, they have their numbers marked in distinctive colours, red for one scale, blue for the next, and so on. *h*, *h*, *h* ... are holes in the disc, to afford a finger purchase in turning it. ECE is a cardboard marker, turning on the axis C, and having its edges radii. Either of these, placed over any point on a scale, indicates the point on any other scale which lies on the same radius.

Without going fully into the theory of the slide rule (for a concise account of which see De Morgan's article in the "Penny Cyclopedia"), it may be as well to remark, that in all circular varieties of that instrument the measurements are angular—the value of an angle, however, being represented by the ratio of its arc to the circumference. The entire circle is taken as unity, but, being visibly identical with n coinciding circles, it also represents any integer n, positive or negative. Hence it represents the characteristic of a logarithm, whatever that may be, $>$ or <0. In the scale *bb* of log log x the characteristic is 0, in *cc* it is -1, and in *dd*, -2. The mantissa, or decimal portion, of a logarithm, being in the tables always a positive quantity <1, alone requires measurement. The measurements are taken from a scale of equal parts constructed round the margin of the rim, but which, in the finished instrument, need scarcely be retained.* They are in the direction of the arrow, those of the disc scales commencing from a common radius *oo′*. The graduation up to *at least* two places of figures is by radial divisions (omitted in fig. 1). Every tenth division, only, has a number attached—that number being a quantity of which the arc is a certain function.

I now come to the diagonal scales, the object of which is to afford

* It might be used in conjunction with *aa* for finding the logarithm of a number, but this can be done by *a′a′* in conjunction with the scales of log log x.

an extra figure in the reading. Where the ordinary slide rule reads only to two figures, by the insertion of these scales between each two consecutive radial divisions, we obtain a third.

Fig. 3 is an enlarged view of two contiguous scales, showing the radial divisions with their diagonal scales and the system of numbering. Each of the rings aa, $a'a'$, bb, &c. (fig. 1) is portioned into ten smaller ones, of uniform breadth, by circles,—the middle circle (corresponding to the figure 5) being described in red. The arcs of these *elementary* circles, as I may call them, lying between consecutive radial divisions, are crossed by a curved diagonal whose equation, of course, depends on the law of graduation of the scale to which it pertains, the number of its intersections with an arc, reckoned from the circles MN, mn, determining the value of the right-hand figure.

Suppose MmnN (fig. 4) to be the space between two consecutive radial divisions; Mn, the diagonal cutting two consecutive elementary arcs in P and Q; and C the common centre. Draw the radii CM, CP, CQ, and CN, and take C for the pole.

Let $r = \mathrm{CP}$, the radius vector;
$\theta =$ *actual* angle from a fixed line;
$\phi =$ angle of reading on the scale;
then we have $\phi = \mathrm{F}(\theta)$ ..... (α),

where F represents the function which the measurement is of the reading on the scale.

Now, although conformable to decimal notation, we insert only nine equidistant arcs between MN and mn, the curve is evidently the same, whatever be their number. Suppose it therefore n, and consider the difference CQ$-$CP of two consecutive radii, Δr, corresponding to the angle PCQ or $\Delta\phi$.

Then $\dfrac{\Delta\phi}{\Delta r} = k$, a constant;

hence, making n infinite, $\dfrac{d\phi}{dr} = k$ ..... (β).

Combining (α) and (β), we obtain, on integration,

$$\mathrm{F}(\theta) = k(r+c),$$

which is the general equation to diagonal curves.

In the scale of equal parts used as a measure in the construction of the other scales, $\mathrm{F}(\theta) = a\theta$;

hence $r+c = \mathrm{A}\theta$

is the equation to its diagonal curve,—the Archimedean spiral.

In the scale of log x,

$$\theta = a \log_{10}\phi, \quad \text{or} \quad \phi = \mathrm{A}10^{\theta},$$

and $r+c = \mathrm{B}10^{\theta}$ is the equation to a diagonal.

In the scale of log log x, it may be shown in a similar manner that

$$r + c = \mathrm{B}\, 10^{10^{\theta}}$$

is the equation.

Of course, in the log x and log log x scales, the constants vary with the position of each diagonal scale in the circumference; and, were we to limit the radial graduation to a two-figure reading, the diagonals at the commencement of each scale would be very long, compared with those near the end, cutting the middle elementary arcs at an angle too acute to afford satisfactory reading, especially when, as might easily occur, there was an appreciable inaccuracy in the tracing of the curve. In order to avoid these long diagonals, the arc MN, when it is more than, say, twice the length of the breadth Mm of the scale, ought to be split into ten portions by the insertion of additional radii, and a diagonal scale constructed on each—the reading, of course, extending to four places of figures.

Instead of all the elementary circles being described on the face of the instrument, short arcs of them might be marked on the margins of the radial marker, the fifth only being shown on the scale.

In my models I have generally substituted, for the true curve of the diagonal, the arc of a circle which would cut it in three points, viz., M, n, and an intermediate point on the fifth elementary circle. Where, however, the diagonals are long, it would be a better plan to determine from tables a greater number of points, and trace the curves in the ordinary way.*

Although I have nowhere seen any description of the diagonal scale as applied to angular measurement, the following statement of Sir D. Brewster, in his "Martyrs of Science," p. 123, which I only came across a few days ago—months after the idea had suggested itself to me—renders it probable that something of the kind was in use in 1576, or even earlier. After giving a list of the astronomical instruments in Tycho Brahe's observatory at Uraniberg, the author remarks:—

"In almost all the instruments now enumerated, the limb was subdivided by diagonal lines,—a method which Tycho first brought into use, but which, in modern times, has been superseded by the inventions of Nonius and Vernier."

Diagonal curves might be advantageously applied to the cylindrical form of slide rule, where the scales are on zones capable of being turned independently of one another round a cylindrical core. A

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* Should the instrument ever become an article of sale, much expense might be avoided by engraving the whole face from one carefully executed copper-plate, and afterwards cutting out the portion which forms the disc. The disc ought to have a metal backing. Probably a more generally useful arrangement than the one described might consist of two log x scales alone, having a face 6 inches in diameter.

marker, having its edge parallel to the axis, must in this case be attached to the instrument.

Suppose OM (fig. 5) to be a portion of one of these scales developed into a plane; O, the point from which the measurements are taken. $ON = x$, $NP = y$, coordinates of a point P on the diagonal Cd.

Then we have $$\frac{dF(x)}{dy} = k;$$

and hence $F(x) = k(y+c)$ is the general equation to these diagonals.

It is easily seen that the diagonals Ab, Bc, Cd, &c., are identical with the arcs ab', $b'c'$, $c'd'$, &c., of the continuous curve O$ab'c'$....., the extension of the first diagonal Oa, and which may be formed by placing the rectangles containing them in the positions shown by dotted lines.

In the scale of log x, $F(x) = a10^x$,

and therefore $y+c = A10^x$ is the equation to Cd. And since, when $x = 0$, $y = 0$, and $c = A$, $y+A = A10^x$ is the equation to O$ab'c'$.

If $A = 1$, this assumes the simpler form,

$$x = \log(1+y).$$

Similarly it may be shown that the equation to a diagonal in the scale of log log x, is of the form

$$y+c = B10^{10^x}.$$

February 11*th*, 1875.

Prof. H. J. S. SMITH, F.R.S., President, in the Chair.

Mr. C. E. Bickmore was elected a Member; and Messrs. A. B. Kempe, B.A., Scholar of Trinity College, Cambridge, and S. A. Renshaw, were proposed for election.

Prof. Cayley, F.R.S., read two short papers: "On a Point in the Theory of Attractions," and on the question of "The Mechanical Description of a Quartic Curve." Prof. Sylvester, F.R.S., exhibited a new sort of Lady's Fan, and briefly indicated its mode of construction and properties. He then spoke in detail on the expression of the curves generated by any given system whatever of link-work under the form of an irreducible determinant. The Chairman made a few remarks on the paper. Mr. Tucker read portions of papers by the Rev. W. H. Laverty, on Peaucellier's Problem; by Mr. Routh, on Laplace's

Three Particles; and by Mr. J. Griffiths, "Note on some relations between certain Elliptic and Hyperbolic Functions."

The following presents were received:—

"Monatsbericht," Sept. and Oct., 1874.

Carte-de-visite of Col. A. R. Clarke, F.R.S., who also gives a photographic likeness of the late Dr. Rutherford, F.R.A.S. (taken by himself), and a photographic copy of a likeness of Prof. Schumacher.

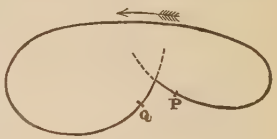
Note on a Point in the Theory of Attraction. By Prof. Cayley.

[*Read February* 11*th*, 1875.]

Consider a mass of matter distributed in any manner on a surface, and attracting points P, Q not on the surface. Consider a point Q accessible from P, viz., such that we can pass continuously from P to Q without passing through the surface. [It is hardly necessary to remark that, if for example the matter is distributed over a hemisphere or segment of a closed surface, then by the surface we mean the hemisphere or segment, not the whole closed surface.] The potential and its differential coefficients *ad infinitum*, in regard to the coordinates of the attracted point, all vary continuously as we pass from P to Q; and it follows that the potential is one and the same analytical function of (a, b, c), the coordinates of the attracted point, for the whole series of points accessible from the original point P; in particular, if the surface be an unclosed surface, for instance a hemisphere or segment of a sphere, then every point Q whatever not on the surface is accessible from P, and the theorem is that the potential is one and the same analytical function of (a, b, c), the coordinates of the attracted point, for any position whatever of this point (not being a point on the surface). But this seems to give rise to a difficulty. Consider the matter as uniformly distributed over a closed surface, and divide the closed surface into two segments: the potential of the whole shell is the sum of the potentials of the two segments; and the potential of the first segment being always one and the same function of (a, b, c), whatever may be the position of the attracted point, and similarly the potential of the second segment being always one and the same function of (a, b, c), whatever may be the position of the attracted point; then the potential of the whole shell is one and the same function of (a, b, c), whatever may be the position of the attracted point. This we know is not the case for a uniform spherical shell; for the potential is a different function for external and interior points, viz., for internal points it is

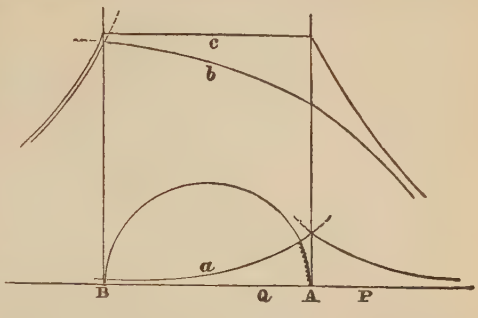
a constant, $= M \div$ radius; for external points it is $= \frac{M}{\sqrt{a^2+b^2+c^2}}$, if a, b, c are the coordinates measured from the centre of the sphere.

The difficulty is rather apparent than real; reverting to the case of an unclosed surface or segment, and considering the continuous curve from P to Q, let this be completed by a curve from Q to P through the segment; viz., we thus have P, Q points on a closed curve or circuit meeting the segment in a single point. To fix the ideas, the circuit may be taken to be a plane curve, and the position of a point on the circuit may be determined by means of its distance s from a fixed point on the circuit. Considering this circuit as drawn on a cylinder, we may at each point of the circuit measure off, say upwards, along the generating line of the cylinder, a length or ordinate z, proportional to the potential of the point on the circuit, the extremities of these distances forming a curve on the cylinder, say the potential curve. We may draw a figure representing this curve only; the points P, Q being marked as if they were points on the curve (viz., at the upper instead of the lower extremities of the corresponding ordinates z), the generating lines of the cylinder, and the plane section which is the circuit, not being shown in the figure. The potential curve is then, as shown in the figure, a continuous curve, viz., we pass from P to Q in the direction of the arrow, or along that part of the circuit which does not meet the segment, a curve without any abrupt change in the value of the ordinate z, or of any of its differential coefficients, $\frac{dz}{ds}$, $\frac{d^2z}{ds^2}$, &c.; but there is, corresponding to the point where the circuit meets the surface, an abrupt change in the direction of the potential curve or value of the differential coefficient $\frac{dz}{ds}$, viz., the point on the curve is really a node, the two branches crossing at an angle, as shown by the dotted lines, but without any potentials corresponding to these dotted lines.



In the case of two segments forming a closed surface, or say two segments forming a complete spherical shell; then, if the points P, Q are one of them internal, the other external, the circuit, assuming it to meet the first segment in one point only, will meet the second segment in at least one point; the potential curves corresponding to the two segments respectively will have each of them, at the point corresponding to the intersection of the circuit with the segment, a node; and it hence appears how, in the potential curve corresponding to the whole shell (for which curve the ordinate z is the sum of the ordinates be-

longing to the two segments respectively), there will be a discontinuity of form corresponding to the passage from an exterior to an interior point. This is best shown by the annexed figure, which represents a uniform spherical shell made up of two segments, one of which is taken to be a small segment or disc having the point A for its centre, the other the large segment B, which is the remainder of the shell; the circuit is taken to be the right line ..PAQB.. through the centre of the sphere (viz., we may imagine the two extremities meeting at infinity, or we may, outside the sphere, bend the line so as to unite the two extremities, thus forming a closed curve). The curve (a) represents the potential curve for the segment A, the curve (b) that for the segment B, these two curves having, as shown by the dotted lines, nodes corresponding to the points A, B respectively (but these dotted portions not indicating any potentials); and then, drawing at each point the ordinate which is the sum of those for the curves (a), (b) respectively, we have the discontinuous curve (c), composed of a horizontal portion and two hyperbolic branches, which is the potential curve for the whole spherical shell.



Practically the figure is constructed by drawing the curves (c), (a), and from them deducing the curve (b); as regards the curve (a) it may be noticed, that treating the segment (a) as a plane disc, the curve (a) is made up of portions of two hyperbolas; viz., it breaks up into two curves, instead of being, as assumed in the discussion, a single curve; this is a mere accident, not affecting the theory; and, in fact, taking the segment to be what it really is, the segment of a sphere, the potential curve does not thus break up.

On the Expression of the Coordinates of a Point of a Quartic Curve as Functions of a Parameter. By Prof. Cayley.

[*Read February 11th*, 1875.]

The present short Note is merely the development of a process of Prof. Sylvester's. It will be recollected that the general quartic curve has the deficiency 3 (or it is 4-cursal); the question is therefore that

of the determination of the subrational* functions of a parameter which have to be considered in the theory of curves of the deficiency 3.

Taking the origin at a point of the curve, the equation is

$$(x, y)^4 + (x, y)^3 + (x, y)^2 + (x, y) = 0;$$

and writing herein $y = \lambda x$, the equation, after throwing out the factor x, becomes

$$(1, \lambda)^4 x^3 + (1, \lambda)^3 x^2 + (1, \lambda)^2 x + (1, \lambda) = 0;$$

or say

$$ax^3 + 3bx^2 + 3cx + d = 0,$$

where we write for shortness

$$a, b, c, d = (1, \lambda)^4, \tfrac{1}{3}(1, \lambda)^3, \tfrac{1}{3}(1, \lambda)^2, (1, \lambda);$$

viz., a, b, c, d stand for functions of λ of the degrees 4, 3, 2, and 1 respectively.

The equation may be written

$$(ax + b)^3 - 3(b^2 - ac)(ax + b) + a^2 d - 3abc + 2b^3 = 0;$$

viz., writing for a moment $ax + b = 2\sqrt{b^2 - ac}\,.\,u$, this is

$$4u^3 - 3u + \frac{a^2 d - 3abc + 2b^3}{2(b^2 - ac)\sqrt{b^2 - ac}} = 0;$$

hence, assuming

$$-\cos\phi = \frac{a^2 d - 3abc + 2b^3}{2(b^2 - ac)\sqrt{b^2 - ac}},$$

then we have $4u^3 - 3u - \cos\phi = 0$, consequently u has the three values $\cos\frac{1}{3}\phi$, $\cos\frac{1}{3}(\phi + 2\pi)$, $\cos\frac{1}{3}(\phi - 2\pi)$; and we may regard $\cos\frac{1}{3}\phi$ as representing any one of these values.

We have thus $ax + b = 2\sqrt{b^2 - ac}\cos\frac{1}{3}\phi$, and $y = \lambda x$, giving x and y as functions of λ and ϕ, that is, of λ. But for their expression in this manner we introduce the irrationality $\sqrt{b^2 - ac}$ [which is of the form $\sqrt{(1, \lambda)^6}$], and the trisection or derivation of $\cos\frac{1}{3}\phi$ from a given value of $\cos\phi$; viz., we have, as above, $-\cos\phi$, a function of λ of the form

$$(1, \lambda)^9 \div (1, \lambda)^6 \sqrt{(1, \lambda)^6}.$$

The equation for ϕ may be expressed in the equivalent forms

$$\sin\phi = \frac{a\sqrt{-(a^2 d^2 + 4ac^3 + 4b^3 d - 6abcd - 3b^2 c^2)}}{(b^2 - ac)\sqrt{b^2 - ac}},$$

$$-\tan\phi = \frac{a\sqrt{-(a^2 d^2 + 4ac^3 + 4b^3 d - 6abcd - 3b^2 c^2)}}{a^2 d - 3abc + 2b^3};$$

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* The expression "subrational" includes irrational, but it is more extensive; if Y, X are rational functions, the same or different, of y, x respectively, and Y is determined as a function of x by an equation of the form $Y = X$, then y is a subrational function of x. The notion is due to Prof. Sylvester.

and inasmuch as we have

$$2\sqrt{b^2-ac} = -\frac{a^2d-3abc+2b^3}{(b^2-ac)\cos\phi},$$

we may, instead of $ax+b = 2\sqrt{b^2-ac}\cos\frac{1}{3}\phi$,

write $$ax+b = -\frac{(a^2d-3abc+2b^3)\cos\frac{1}{3}\phi}{(b^2-ac)\cos\phi};$$

or, what is the same thing,

$$= \frac{-(a^2d-3abc+2b^3)}{(b^2-ac)(4\cos^2\frac{1}{3}\varphi-3)}.$$

The formulæ may be simplified by introducing μ, a function of λ, determined by the equation $c\mu^2-2b\mu+a=0$;
viz., this equation is $\frac{1}{3}(1,\lambda)^2\mu^2-\frac{2}{3}(1,\lambda)^3\mu+(1,\lambda)^4=0$, so that (λ,μ) may be regarded as coordinates of a point on a nodal quartic curve, or quartic curve of the next inferior deficiency 2. And we then have

$$(c\mu-b) = \sqrt{b^2-ac},$$

and consequently $$-\cos\phi = \frac{a^2d-3abc+2b^3}{2(c\mu-b)^3};$$

viz., $\cos\phi$ is given as a rational function of the coordinates (λ,μ); there is, as before, the trisection; and we then have

$$ax+b = 2(c\mu-b)\cos\tfrac{1}{3}\phi, \quad y=\lambda x,$$

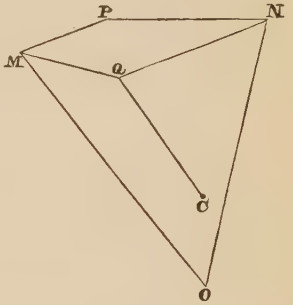
giving x and y as functions of λ, μ, ϕ; that is, ultimately, as functions of λ. I have not succeeded in obtaining in a good geometrical form the relation between the point (x,y) on the given quartic and the point (λ,μ) on the nodal quartic.

Reverting to the expression of $\tan\phi$, it may be remarked that $a=0$ gives the values of λ which correspond to the four points at infinity on the given quartic curve; $a^2d^2+4ac^3+4b^3d-6abcd-3b^2c^2=0$, the values corresponding to the ten tangents from the origin; and $a^2d-3abc+2b^3=0$, the values corresponding to the nine lines through the origin, which are each such that the origin is the centre of gravity of the other three points on the line.

I take the opportunity of mentioning a mechanical construction of the Cartesian. The equation $r'=-\mathrm{A}\cos\theta-\mathrm{N}$ represents a limaçon (which is derivable mechanically from the circle $r'=-\mathrm{A}\cos\theta$), and if we effect the transformation $r'=r+\frac{\mathrm{B}}{r}$, the new curve is $r+\frac{\mathrm{B}}{r}+\mathrm{A}\cos\theta+\mathrm{N}=0$; that is, $r^2+r(\mathrm{A}\cos\theta+\mathrm{N})+\mathrm{B}=0$, which is, in fact, the equation of a Cartesian. The assumed transformation $r'=r+\frac{\mathrm{B}}{r}$ can be effected immediately by a Peaucellier cell.

Extension of Peaucellier's Theorem. *By* W. H. Laverty.

[*Read February 11th*, 1875.]



The general problem is—"Given O and C fixed, and the lengths of OM, ON, CQ, QM, QN, MP, and NP; to determine the locus of P;" but a great simplification is obviously introduced by taking MP = MQ and NP = NQ; for then part of the locus of P coincides with the locus of Q, and a factor divides out of the equation to the locus of P.

Take O the origin of rectangular coordinates, OC the axis of y, and let the coordinates of C, M, N, Q, P be respectively $0, \beta$; x_1, y_1; x_2, y_2; x', y'; x, y; then the problem is this—"To eliminate $x'y'x_1y_1x_2y_2$ between the seven equations,

$$x'^2+(y'-\beta)^2=a^2 \quad \ldots\ldots (1),$$

$$x_1^2+y_1^2=r_1^2 \quad \ldots\ldots (2),$$

$$x_2^2+y_2^2=r_2^2 \quad \ldots\ldots (3),$$

$$(x'-x_1)^2+(y'-y_1)^2=\rho_1^2 \quad \ldots\ldots (4),$$

$$(x-x_1)^2+(y-y_1)^2=\rho_1^2 \quad \ldots\ldots (5),$$

$$(x'-x_2)^2+(y'-y_2)^2=\rho_2^2 \quad \ldots\ldots (6),$$

$$(x-x_2)^2+(y-y_2)^2=\rho_2^2 \quad \ldots\ldots (7);$$

a, r_1, r_2, ρ_1, and ρ_2 being respectively CQ, OM, ON, MQ (= MP), and NQ (= NP).

The following abbreviations will be found convenient:—

$$\delta_1^2=\rho_1^2-r_1^2+\beta^2-a^2,$$

$$\delta_2^2=\rho_2^2-r_2^2+\beta^2-a^2,$$

$$\lambda^2=\delta_1^2-\delta_2^2,$$

$$M_1^2=x^2+y^2+r_1^2-\rho_1^2,$$

$$M_2^2=x^2+y^2+r_2^2-\rho_2^2;$$

therefore

$$M_2^2-M_1^2=\lambda^2,$$

$$\left.\begin{aligned} K^2 &= x^2+(y-\beta)^2-a^2 \\ &= M_1^2+\delta_1^2-2\beta\gamma \\ &= M_2^2+\delta_2^2-2\beta\gamma \end{aligned}\right\} \quad \ldots\ldots (8).$$

[K^2, obviously, is the factor which is to divide out of the equation.]

$$R_1^2 = (x^2+y^2)-(r_1-\rho_1)^2,$$
$$R_2^2 = (x^2+y^2)-(r_2-\rho_2)^2,$$
$$S_1^2 = (r_1+\rho_1)^2-(x^2+y^2),$$
$$S_2^2 = (r_2+\rho_2)^2-(x^2+y^2),$$
$$A^2 = R_1S_1-R_2S_2,$$
$$B^4 = \delta_2^2 R_1 S_1-\delta_1^2 R_2 S_2,$$
$$D^4 = B^4-2\lambda^2\beta x.$$

Now, from equations (2) and (5) find x_1 and y_1, and from equations (3) and (7), x_2 and y_2, in terms of x and y. Then from equations (4) and (6) determine x' and y' in terms of x_1, y_1, x_2, and y_2; this, by substitution, gives the values of x' and y' in terms of x and y, substituting which values in equation (1), and substituting for M_1^2 and M_2^2 from equation (8), we have, as the equation to the locus of P,

$$a^2[K^2A^2-D^4]^2 = [yK^2\lambda^2+xD^4]^2+[K^2(x\lambda^2-\beta A^2)-(y-\beta)D^4]^2.$$

This equation is to be divided through by K^2; and all the terms are separately divisible by K^2, except the various coefficients of D^8; the sum of these, however, being

$$-a^2+x^2+(y-\beta)^2,$$

is itself K^2; therefore, dividing by K^2, the locus becomes finally

$$a^2A^2(K^2A^2-2D^4) = y\lambda^2(K^2y\lambda^2+2xD^4)$$
$$+(x\lambda^2-\beta A^2)\{K^2(x\lambda^2-\beta A^2)-2(y-\beta)D^4\}+D^8;$$

or, more conveniently,

$$A^2\{(K^2A^2-2D^4)(a^2-\beta^2)-2\beta yD^4\}-D^8$$
$$=\lambda^2(A^2K^2+D^4)2\beta x+\lambda^4(x^2+y^2)K^2.$$

As a particular case take $\delta_1^2 = \delta_2^2 = \delta^2$;

then $D^4 = A^2\delta^2$,

for $\lambda^2 = \delta_1^2-\delta_2^2 = 0$;

and the general equation reduces to

$$(K^2-2\delta^2)(a^2-\beta^2)-2\beta y\delta^2 = \delta^4,$$

which is the equation to a circle with its centre on the line joining the fixed points; the circle degenerating into a straight line perpendicular to the fixed line when $a^2 = \beta^2$; *i.e.*, when $CQ = CO$.

The result arrived at is, therefore, an extension of Peaucellier's Theorem. This last proves that, when $OM = ON$ and $MQ = MP = NQ = NP$, then the locus of P is a circle degenerating into a straight line when $CQ = CO$. By the present paper a similar result is shewn to be true when the links are bound by the more general conditions

$$MP = MQ, \quad NP = NQ, \quad \text{and} \quad OM^2-MQ^2 = ON^2-NQ^2.$$

On Laplace's Three Particles, with a Supplement on the Stability of Steady Motion. *By* E. J. Routh, F.R.S.

[*Read February* 11*th*, 1875.]

Laplace has shewn that, if three particles be placed at the corners of an equilateral triangle and properly projected, they will move under their mutual attractions so as always to remain at the angular points of an equilateral triangle. He also shewed that, if they be placed in a straight line, and properly projected, they will always remain in the same straight line. He remarks that, if the earth and moon had been originally placed in the same straight line with the sun, at distances from the sun proportional to 1 and $1+\frac{1}{100}$, and properly projected, the moon would always have been in opposition to the sun. The moon would have been too distant to have been in a state of continual eclipse, and thus would have been full every night. But Liouville has shewn, in the "Additions à la Connaissance des Temps," that such a motion is unstable.

The question arises, whether the motion when the particles are at the angular points of an equilateral triangle, is also unstable. I find, by a brief note in Jullien's Problems, that this question has been discussed by M. Gascheau in a Thèse de Mécanique, who has arrived at the result that the motion is stable when the square of the sum of the masses is greater than 27 times the sum of the products of the masses taken two and two together. But no reference to where M. Gascheau's work can be found is given, and I have been unable to discover it.

I therefore proceeded to investigate the motion on the supposition that the law of attraction is the inverse κ^{th} power of the distance. I found the following results:—

1. The motion cannot be stable unless κ is less than 3.

2. The motion is stable, whatever the masses may be, if the law of force be expressed by any positive power of the distance, or any negative power less than unity. For other powers the stability will depend on the relation between the masses.

3. The motion is stable to a first approximation if

$$\frac{(\mathrm{M}+m+m')^2}{\mathrm{M}m+\mathrm{M}m'+mm'} > 3\left(\frac{1+\kappa}{3-\kappa}\right)^2,$$

where M, m, m' are the masses. This agrees with M. Gascheau's result if $\kappa=2$, or the law of force be the law of nature.

4. When two of the masses are much smaller than the third, the inequalities in their angular distances, as seen from the large body, have a much greater coefficient than their linear distances from the same body.

5. On proceeding to a second approximation it would seem that the form of the triangle joining the three particles is very little altered by

any disturbance, but in certain cases, depending on the nature of the disturbance, the size of the triangle may be subject to very considerable variations.

As a supplement, I have generalized the reasoning of the problem of the three bodies, so as to obtain the form of the determinantal equation, to find the periods of oscillation of any dynamical system about a state of steady motion in which the *vis viva* is constant. Two limitations have been made; first, the system must be under a conservative system of forces; and secondly, the *vis viva* can be expressed in terms of the coordinates so as not to contain the time explicitly. The equation is then shown to be always of an *even* order, and the condition of stability is that all the roots should be real and negative.

The three particles, when in motion, remain always very nearly at the angular points of an equilateral triangle. The triangles formed by joining the particles to their common centre of gravity are not marked by any simplicity of form. Instead of referring the motion to the centre of gravity, it will be more convenient to reduce one of the particles to rest, and to consider the relative motion of the other two. We have thus only one triangle to examine, and that one nearly equilateral.

Let the mass M of the particle to be reduced to rest be taken as unity, and let m, m' be the masses of the other two. Let the distances Mm, Mm' be respectively $a+x$ and $a+\xi$, where x and ξ are small; and let the angle made by these radii vectores be $\frac{\pi}{3}+\eta$. Let the angle the radius vector joining M, m makes with some fixed line in space, be $nt+y$, where y is a small quantity.

The accelerations along the radius vector, and perpendicular to it, are

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = \frac{d^2x}{dt^2} - (a+x)\left(n+\frac{dy}{dt}\right)^2$$

$$= \frac{d^2x}{dt^2} - n^2x - 2an\frac{dy}{dt} - an^2,$$

$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 2n\frac{dx}{dt} + a\frac{d^2y}{dt^2}.$$

To express the forces on the particles, let r, r', R be the distances between the particles Mm, Mm', mm'; and let ϕ', ϕ, ψ be the angles opposite to these distances. Let the law of attraction be the inverse κ^{th} power.

The equations of motion are

$$\left.\begin{aligned} \frac{d^2x}{dt^2} - n^2x - 2an\frac{dy}{dt} - an^2 + \frac{1+m}{r^\kappa} + \frac{m'\cos\psi}{r'^\kappa} + \frac{m'\cos\phi}{\mathrm{R}^\kappa} = 0 \\ 2n\frac{dx}{dt} + a\frac{d^2y}{dt^2} + \frac{m'\sin\psi}{r'^\kappa} - \frac{m'\sin\phi}{\mathrm{R}^\kappa} = 0 \end{aligned}\right\},$$

$$\left.\begin{aligned}\frac{d^2\xi}{dt^2}-n^2\xi-2an\frac{d(y+\eta)}{dt}-an^2+\frac{1+m'}{r'^\kappa}+\frac{m\cos\psi}{r^\kappa}+\frac{m\cos\phi'}{\mathrm{R}^\kappa}=0\\ 2n\frac{d\xi}{dt}+a\frac{d^2(y+\eta)}{dt^2}-\frac{m\sin\psi}{r^\kappa}+m\frac{\sin\phi'}{\mathrm{R}^\kappa}=0\end{aligned}\right\}.$$

Putting x, y, ξ, η all zero, we have in *steady motion*

$$a^{\kappa+1}n^2=1+m+m';$$

let us put $a^{\kappa+1}=b$ for brevity.

We have now to calculate these forces

$$\begin{aligned}\mathrm{R}^2&=(a+x)^2+(a+\xi)^2-2(a+x)(a+\xi)\cos\left(\frac{\pi}{3}+\eta\right)\\&=a^2+ax+a\xi+a^2\sqrt{3}\,\eta;\end{aligned}$$

therefore

$$\mathrm{R}=a+\frac{x+\xi}{2}+\frac{\sqrt{3}}{2}a\eta.$$

Let us write this

$$\mathrm{R}=a+\rho.$$

Let $\phi=\frac{\pi}{3}+\theta$, then we have

$$\sin\left(\frac{\pi}{3}+\theta\right)=\sin\left(\frac{\pi}{3}+\eta\right)\frac{a+\xi}{\mathrm{R}};$$

expanding and rejecting squares of small quantities,

$$\theta=\eta+\sqrt{3}\,\frac{\xi-\rho}{a}.$$

Similarly, if $\phi'=\frac{\pi}{3}+\theta'$, we have

$$\theta'=\eta+\sqrt{3}\,\frac{x-\rho}{a}.$$

We now have

$$\begin{aligned}\frac{\sin\phi}{\mathrm{R}^\kappa}&=\frac{\sqrt{3}}{2}\frac{1}{a^\kappa}\left(1+\frac{\theta}{\sqrt{3}}\right)\left(1-\kappa\frac{\rho}{a}\right)\\&=\frac{\sqrt{3}}{2}\frac{1}{a^\kappa}\left\{1-\frac{3\kappa+1}{2\sqrt{3}}\eta-\frac{\kappa+1}{2}\frac{x}{a}-\frac{\kappa-1}{2}\frac{\xi}{a}\right\}.\end{aligned}$$

Similarly, interchanging x, ξ,

$$\frac{\sin\phi'}{\mathrm{R}^\kappa}=\frac{\sqrt{3}}{2}\frac{1}{a^\kappa}\left\{1-\frac{3\kappa+1}{2\sqrt{3}}\eta-\frac{\kappa+1}{2}\frac{\xi}{a}-\frac{\kappa-1}{2}\frac{x}{a}\right\},$$

$$\begin{aligned}\frac{\cos\phi}{\mathrm{R}^\kappa}&=\frac{1}{2}\frac{1}{a^\kappa}(1-\sqrt{3}\,\theta)\left(1-\kappa\frac{\rho}{a}\right)\\&=\frac{1}{2}\frac{1}{a^\kappa}\left\{1-\sqrt{3}\,\frac{\kappa-1}{2}\eta-\frac{\kappa-3}{2}\frac{x}{a}-\frac{\kappa+3}{2}\frac{\xi}{a}\right\}.\end{aligned}$$

Similarly,

$$\frac{\cos\phi'}{\mathrm{R}^\kappa}=\frac{1}{2}\frac{1}{a^\kappa}\left\{1-\sqrt{3}\,\frac{\kappa-1}{2}\eta-\frac{\kappa-3}{2}\frac{\xi}{a}-\frac{\kappa+3}{2}\frac{x}{a}\right\};$$

also

$$\frac{\cos\psi}{r^\kappa} = \frac{1}{2}\frac{1}{a^\kappa}\left(1-\sqrt{3}\,\eta-\kappa\frac{x}{a}\right),$$

$$\frac{\cos\psi}{r'^\kappa} = \frac{1}{2}\frac{1}{a^\kappa}\left(1-\sqrt{3}\,\eta-\kappa\frac{\xi}{a}\right),$$

$$\frac{\sin\psi}{r^\kappa} = \frac{\sqrt{3}}{2}\frac{1}{a^\kappa}\left(1+\frac{\eta}{\sqrt{3}}-\kappa\frac{x}{a}\right),$$

$$\frac{\sin\psi}{r'^\kappa} = \frac{\sqrt{3}}{2}\frac{1}{a^\kappa}\left(1+\frac{\eta}{\sqrt{3}}-\kappa\frac{\xi}{a}\right).$$

Substituting, the four equations become, when D is written for $\frac{d}{dt}$.

$$\left\{bD^2-(\kappa+1)\left(1+m+\frac{m'}{4}\right)\right\}x-2abnDy$$
$$-m'\frac{\sqrt{3}}{4}(\kappa+1)\,a\eta-\tfrac{3}{4}m'(\kappa+1)\,\xi=0,$$

$$\left\{2bnD+m'\frac{\sqrt{3}}{4}(\kappa+1)\right\}x+abD^2y$$
$$+m'\tfrac{3}{4}(\kappa+1)\,a\eta-\frac{\sqrt{3}}{4}m'(\kappa+1)\,\xi=0,$$

$$\left\{bD^2-(\kappa+1)\left(1+\frac{m}{4}+m'\right)\right\}\xi-2abnDy$$
$$-\left\{2abnD+m\frac{\sqrt{3}}{4}(\kappa+1)\,a\right\}\eta-\tfrac{3}{4}m(\kappa+1)\,x=0,$$

$$\left\{-2bnD+m\frac{\sqrt{3}}{4}(\kappa+1)\right\}\xi-abD^2y$$
$$+\left\{-abD^2+m\tfrac{3}{4}(\kappa+1)\,a\right\}\eta-m\frac{\sqrt{3}}{4}(\kappa+1)\,x=0.$$

Let us now write $\xi=x+X$. Then a variation of x, y alone, without X, η, will represent a variation of steady motion in which the particles always keep at the sides of an equilateral triangle, while a variation of X, η will represent a change from the equilateral form. One object of this transformation is, that the final determinant will have one of its roots obvious previous to expansion, and thus a great amount of numerical labour will be avoided. The four equations may be conveniently written

x	y	X	η
$bD^2-(\kappa+1)(1+m+m')$	$-2abnD$	$-\frac{3}{4}m'(\kappa+1)$	$-\frac{\sqrt{3}}{4}m'(\kappa+1)a$
$2bnD$	abD^2	$-\frac{\sqrt{3}}{4}m'(\kappa+1)$	$\frac{3}{4}m'(\kappa+1)a$
$bD^2-(\kappa+1)(1+m+m')$	$-2abnD$	$bD^2-(\kappa+1)\left(1+\frac{m}{4}+m'\right)$	$-2abnD-\frac{\sqrt{3}}{4}m(\kappa+1)a$
$2bnD$	abD^2	$2bnD-\frac{\sqrt{3}}{4}(\kappa+1)\,m$	$abD^2-\frac{3}{4}m(\kappa+1)a$

$=0.$

To solve these equations, we put

$$x = Ae^{\lambda t}, \quad y = Be^{\lambda t}, \quad X = Ce^{\lambda t}, \quad \eta = De^{\lambda t};$$

we evidently obtain the same determinant with λ written for D. This is the equation to find λ.

We see at once that one factor is λ. This also follows from the physical consideration that a change of y without variation of x, X, or η is a possible motion.

Again, we know that a variation of x and y together without X, η is another possible motion; accordingly subtract from the x column $2n$ times the y column, these two columns then become

x	y
$b\lambda^2-(\kappa-3)(1+m+m')$	$-2bn$
0	$b\lambda$
$b\lambda^2-(\kappa-3)(1+m+m')$	$-2bn$
0	$b\lambda$

Clearly we have as one factor

$$b\lambda^2-(\kappa-3)(1+m+m')=0.$$

In order that the coordinates x, y, X, η may have only periodic terms, we must have κ less than 3.

To find the other factors, we divide the determinant by the factors already found. Let us also subtract the first row from the third, and the second from the fourth. We now have

$$\begin{vmatrix} 1 & \&c. & \&c. & \&c. \\ 0 & b\lambda & \&c. & \&c. \\ 0 & 0 & b\lambda^2-(\kappa+1)\left(1+\frac{m}{4}+\frac{m'}{4}\right) & -2abn\lambda-\frac{\sqrt{3}}{4}(\kappa+1)(m-m')\,a \\ 0 & 0 & 2bn\lambda-\frac{\sqrt{3}}{4}(\kappa+1)(m-m') & ab\lambda^2-\frac{3}{4}(\kappa+1)(m+m')\,a \end{vmatrix} = 0.$$

The expansion is now easy. We see that there is another factor λ.

Also $b^2\lambda^4+b\lambda^2(3-\kappa)(1+m+m')+\frac{3}{4}(1+\kappa)^2(m+m'+mm')=0$;

therefore $$b\lambda^2=-\frac{3-\kappa}{2}(1+m+m')$$
$$\pm\tfrac{1}{2}\sqrt{(1+m+m')^2(3-\kappa)^2-3(m+m'+mm')(1+\kappa)^2}.$$

If, then, the quantity under the root be positive, and also κ less than 3, all the values of λ^2 are real and negative. In this case the motion is stable. If the quantity under the root be negative, we have λ of the form $\pm(\alpha\pm\beta\sqrt{-1})$, and hence the coordinates will have both a positive and a negative real exponent. In this case the motion is unstable.

Since $$(1+m+m')^2 > 3(m+m'+mm'),$$
we see that the motion is stable for all masses if
$(3-\kappa)^2 > (1-\kappa)^2$, or $(3-\kappa+1+\kappa)(3-\kappa-1-\kappa) > 0$; *i.e.*, if $1-\kappa > 0$.
The steady motion is therefore stable for any law of force expressed by a positive power of the distance, and for any law expressed by a negative power less than unity, whatever the masses may be. For other powers the stability will depend on the relation between the masses.

To express the coordinates in terms of the time, we must return to the equations. Replacing the λ by D, they may be written in the form

x	y	X	η	
$b\mathrm{D}^2-(\kappa+1)bn^2$	$-2abn\mathrm{D}$	$-\frac{3}{4}m'(\kappa+1)$	$-\frac{\sqrt{3}}{4}m'(\kappa+1)a$	
$2bn\mathrm{D}$	$ab\mathrm{D}^2$	$-\frac{\sqrt{3}}{4}m'(\kappa+1)$	$\frac{3}{4}m'(\kappa+1)a$	
0	0	$b\mathrm{D}^2-\left(1+\frac{m}{4}+\frac{m'}{4}\right)(\kappa+1)$	$-2abn\mathrm{D}-\frac{\sqrt{3}}{4}(m-m')(\kappa+1)a$	$=0.$
0	0	$2bn\mathrm{D}-\frac{\sqrt{3}}{4}(m-m')(\kappa+1)$	$ab\mathrm{D}^2-\frac{3}{4}(m+m')(\kappa+1)a$	

The roots $\lambda^2=0$ give
$$x = \mathrm{A}_1+\mathrm{A}_2 t, \qquad \mathrm{X} = \mathrm{C}_1+\mathrm{C}_2 t,$$
$$y = \mathrm{B}_1+\mathrm{B}_2 t, \qquad \eta = \mathrm{D}_1+\mathrm{D}_2 t;$$
we see at once that the C's and D's are zero. Also the second line gives $\mathrm{A}_2=0$, and the first $\mathrm{B}_2 = -\frac{\kappa+1}{2}\frac{\mathrm{A}_1}{a}n$.

The roots $\lambda^2 = -(3-\kappa)n^2$ give, if $\mu^2 = -\lambda^2$,
$$x = \mathrm{A}\cos(\mu t+\mu'),$$
$$y = \mathrm{B}\sin(\mu t+\mu'),$$
$$\mathrm{X} = \mathrm{C}_1\cos(\mu t+\mu')+\mathrm{C}_2\sin(\mu t+\mu'),$$
$$\eta = \mathrm{D}_1\cos(\mu t+\mu')+\mathrm{D}_2\sin(\mu t+\mu').$$
The last two lines give (since these roots can be equal to the next pair only when $\kappa=-1$) all the C's and D's zero, and the second line gives $\mathrm{B} = -\frac{2n}{a\mu}\mathrm{A}$, except when $\kappa=-1$.

Lastly, let $-\mu^2$ and $-\nu^2$ be the values of λ^2 with the radical
$$\mathrm{X} = \mathrm{C}_1\cos(\mu t+\mu')+\mathrm{C}_2\cos(\nu t+\nu').$$
Taking the first of these as a type, we get from the fourth line,
$$\eta = \frac{2bn\mu}{\Delta}\mathrm{C}\sin(\mu t+\mu') + \frac{\frac{\sqrt{3}}{4}(\kappa+1)(m-m')}{\Delta}\mathrm{C}\cos(\mu t+\mu'),$$
where $$\Delta = ab\mu^2+\tfrac{3}{4}(\kappa+1)(m+m')a.$$

Differentiating the first line, and adding to it $2n$ times the second, we get

$$b\{D^3-(\kappa-3)n^2D\}x=\frac{\sqrt{3}}{4}m'(\kappa+1)\{(\sqrt{3}D+2n)X+(D-2n\sqrt{3})a\eta\};$$

therefore $$x=P\sin(\mu t+\mu')+Q\cos(\mu t+\mu'),$$

where $$P=\frac{-4an\{b\mu^2+\frac{3}{4}(\kappa+1)m'\}}{\Delta\Delta'\mu}\cdot\frac{\sqrt{3}}{4}m'(\kappa+1),$$

$$Q=\frac{-\sqrt{3}\,a\{b(\mu^2-4n^2)+\frac{1}{2}(\kappa+1)(m+2m')\}}{\Delta\Delta'}\cdot\frac{\sqrt{3}}{4}m'(\kappa+1),$$

where $$\Delta'=b\{\mu^2+(\kappa-3)n^2\}.$$

In the case in which two of the particles m, m' are much smaller than the third, these results admit of some simplification. Neglecting the squares of m, m', we find

$$b\mu^2=\frac{3}{4}\frac{(1+\kappa)^2}{3-\kappa}(m+m'),$$

$$b\nu^2=(3-\kappa)bn^2-\frac{3}{4}\frac{(1+\kappa)^2}{3-\kappa}(m+m').$$

Taking the former, we have

$$\Delta=3\frac{1+\kappa}{3-\kappa}(m+m')a=\frac{4}{1+\kappa}ab\mu^2,$$

which is small. We then have $X=C\cos\mu t$,

$$\eta=\frac{1+\kappa}{2}\frac{n}{\mu}\frac{C}{a}\sin\mu t+\frac{3-\kappa}{4\sqrt{3}}\frac{m-m'}{m+m'}\frac{C}{a}\cos\mu t,$$

$$x=\frac{\sqrt{3}}{4}(1+\kappa)\frac{m'}{bn\mu}\frac{(1+\kappa)m+4m'}{m+m'}C\sin\mu t-\frac{m'}{m+m'}C\cos\mu t.$$

Since m' is of the order μ^2, we see that the most important term is

$$\eta=\frac{1+\kappa}{2}\frac{n}{\mu}\frac{C}{a}\sin\mu t,$$

so that there is a large inequality in the angular distance between the two smaller particles, while their distances from the central larger body are very little altered.

Taking the other value of μ, we have

$$\Delta'=-\frac{3}{4}\frac{(1+\kappa)^2}{3-\kappa}(m+m'),$$

which is small. We then have $X=C\cos\mu t$,

$$\eta=\frac{2n}{\mu}\frac{C}{a}\sin\mu t+\frac{\sqrt{3}}{4}\frac{1+\kappa}{3-\kappa}(m-m')\frac{C}{a}\cos\mu t,$$

$$x=\frac{4}{\sqrt{3}}\frac{3-\kappa}{1+\kappa}\frac{n}{\mu}\frac{m'}{m+m'}C\sin\mu t-\frac{m'}{m+m'}C\cos\mu t,$$

none of these terms rise to any importance.

The values of the coordinates may be written

$$x = \alpha + A_1 \cos(n\sqrt{3-\kappa}\, t + \epsilon) + A_2 \cos \mu t + A_3 \sin \mu t + A_4 \cos \nu t + A_5 \sin \nu t,$$

$$y = \beta - \frac{\kappa+1}{2}\frac{a}{a} nt - \frac{2A_1}{a\sqrt{3-\kappa}} \sin(n\sqrt{3-\kappa}\, t + \epsilon) + B_2 \cos \mu t + \&c.,$$

$$X = C_2 \cos \mu t + \&c.,$$

$$\eta = D_2 \cos \mu t + \&c.$$

Since x is merely a small addition to a, we may suppose the α to be included in the a. In this case the corresponding term in y, containing t, is absorbed in the n. These two terms may therefore be omitted.

The case $\kappa = -1$ has been excepted. The law of attraction is then "directly as the distance." As this case has no special interest, we shall not further consider it.

Though the motion may be stable to a first approximation, yet it is important to examine the terms of the second order to ascertain if there be any which may ultimately disturb this stability.

If we repeat our calculation of the impressed forces, retaining now the squares and products of the small quantities, we shall evidently have additional terms of the form

$$E_1 x^2 + E_2 X^2 + E_3 \eta^2 + 2E_4 xX + 2E_5 x\eta + 2E_6 X\eta$$

in each of the four equations of motion. We must also add to the effective forces the terms

$$-a\left(\frac{dy}{dt}\right)^2 - 2nx\frac{dy}{dt}, \quad 2\frac{dx}{dt}\frac{dy}{dt} + x\frac{d^2y}{dt^2}$$

in the first two equations respectively, and similar expressions with $x+X$ and $y+\eta$ written for x, y in the two last equations.

We have now to substitute in these expressions of the second order the first approximate values of x, X, and η. We shall clearly obtain in every case an expression of the form

constant + periodic terms.

The periodic terms will none of them have the same period as those in the first approximation; because their arguments are the sums and differences of those of the first approximation, and it is easily seen that one argument cannot generally be double the other. Each periodic term will therefore give rise to a corresponding periodic term of the same period as itself. These therefore cannot affect the stability of the motion.

Thus the three roots are $b\lambda^2 = (\kappa-3)(1+m+m')$, $b\mu^2$, and $b\nu^2$. If $\lambda^2 = 4\mu^2$, or $4\nu^2$, we have $\mu^2 = \nu^2$, and the motion is unstable. If $\mu^2 = 4\lambda^2$, we get, by substituting in the equation,

$$12(\kappa-3)^2(1+m+m')^2 + \tfrac{3}{4}(1+\kappa)^2(m+m'+mm') = 0,$$

which is impossible. And if $\mu^2 = 4\nu^3$, we have

$$\frac{(1+m+m')^2}{m+m'+mm'} = \frac{25}{16} \cdot 3 \cdot \frac{(1+\kappa)^2}{(3-\kappa)^2},$$

which is possible. When this equality holds very nearly between the masses, there will be a periodic term in the disturbance which will rise from the second order to the first. If the equality hold exactly, the nature of the motion will be different from that taken as the first approximation.

Next let us consider the effect of the constant terms. Referring to the diagram on page 91, we form the equations by writing some constants e, f, g, h on the right-hand sides instead of zero. The two last equations contain only X and η, and we see that they may be satisfied by giving X and η two additional constant terms. Both of these are of the second order of small quantities, because their denominator, being of the form $\frac{m+m'+mm'}{\alpha m+\beta m'}$, cannot be small. The effect of these constant terms is not sensibly to alter the configuration of the particles; *i.e.*, they will still be very nearly at the angular points of an equilateral triangle.

The two first equations, on substituting for X and η their constant values just found, take the form

$$\begin{array}{cc} x & y \\ \left|\begin{array}{cc} b\mathrm{D}^2-(\kappa+1)\,bn^2 & -2abn\mathrm{D} \\ 2bn\mathrm{D} & ab\mathrm{D}^2 \end{array}\right| & = \left|\begin{array}{c} e' \\ f' \end{array}\right| \end{array}$$

These give a term $\frac{2f't}{(3-\kappa)\,bn}$ in the value of x, and a term $-\frac{1+\kappa}{3-\kappa}\frac{f'}{ab}\frac{t^2}{2}$ in the value of y. When t becomes sufficiently great to make these terms perceptible notwithstanding the smallness of f', the motion will begin to change from the character given by the first approximation. The equilateral triangle will alter in size, so that we cannot any longer consider x and y to be small.

We know, by Laplace's demonstration that the motion represented by the single type $\cos\sqrt{3-\kappa}\,nt$ must be stable. Thus, if the force be the inverse square, each of the particles m, m' will describe ellipses round M, and will always form an equilateral triangle. Let us examine what the constants e, f, g, h become when X and η are both zero. In this case the impressed forces are functions of x only, and are easily calculated. Putting $x = \mathrm{A}\cos\lambda t$, $y = \mathrm{B}\sin\lambda t$, we find the constant terms to be added to the left sides of the four equations in page 89, to be respectively

$$-a\frac{B^2}{2}\lambda^2-AB n\lambda+\kappa\frac{\kappa+1}{4}(1+m+m')A^2,$$

zero,

$$-a\frac{B^2}{2}\lambda^3-AB n\lambda+\kappa\frac{\kappa+1}{4}(1+m+m')A^2,$$

zero.

Hence the constants e, f, g, h on the right-hand side of the equations on page 91 become $f=0$, $g=0$, $h=0$; so that X and η have no constant additions. Also $f'=0$, and x and y, have only periodic terms, and therefore will remain small.

But this is not the case for variations of the type $\cos\mu t$ and $\cos\nu t$. The calculation of the constants e, f, g, h is very laborious. It is true that the constant terms can only arise from the product of two cosines, or two sines of the same angle, so that we may take the corresponding terms separately. But even with this simplification the arithmetic is very considerable. The want of sufficient leisure to effect this obliges me to leave the determination of these terms for a future opportunity.

Supplement.

Each of the three particles in the state of steady motion is describing a circle, so that their *vis viva* is constant, and the problem just considered was the determination of their times of oscillation. In the same way, if a dynamical system of any kind be making small oscillations about a state of steady motion in which the *vis viva* is constant, we may, by help of Lagrange's Equations, write down the determinantal equation to find the periods of oscillation.

Let the system be referred to coordinates x, y, z, &c., and be oscillating about a mean state of motion. The general value of the *vis viva* 2T is given by $2T = Ax'^2+2Bx'y'+\ldots\ldots$, where A, B, &c. are functions of x, y, z, &c. Let us now put $x=\alpha+mt+\theta$, $y=\beta+nt+\phi$, &c., where θ, ϕ, &c., will be considered small, the mean motion being given by the coordinates when $\theta=0$, $\phi=0$, &c. Then we may expand T in the form

$$\begin{aligned}T = {} & T_0+T_\theta\theta+T_\phi\phi+\ldots\\ & +T_{\theta'}\theta'+T_{\phi'}\phi'+\ldots\\ & +(T_{\theta\theta}\theta^2+2T_{\theta\phi}\theta\phi+\ldots)\tfrac{1}{2}\\ & +T_{\theta\theta'}\theta\theta'+T_{\theta\phi'}\theta\phi'+\ldots\\ & +(T_{\theta'\theta'}\theta'^2+2T_{\theta'\phi'}\theta'\phi'+\ldots)\tfrac{1}{2}.\end{aligned}$$

We may make a similar expansion for the potential U of the forces

$$\begin{aligned}U = {} & U_0+u_\theta\theta+u_\phi\phi+\ldots\\ & +(u_{\theta\theta}\theta^2+2u_{\theta\phi}\theta\phi+\ldots)\tfrac{1}{2}.\end{aligned}$$

We shall now suppose that all the coefficients are independent of t.

(1) This will happen if only some of the coordinates vary in the state of steady motion, and if the coordinates A, B, &c., in the expression for 2T, be independent of these. This is the case which generally occurs, for example in the top problem. Thus, if the steady motion be given by $x = \alpha + mt$, $y = \beta + nt$, $z = \gamma$, then, if x and y be absent from T, the coefficients T_θ, &c. will all be constants.

(2) Even if the coefficients be functions of x, y, z, yet the relations between the quantities m, n, p, &c, given by the equations of steady motion, may cause t to disappear from the coefficients. The effect of this supposition is, that the mean motion is what is usually called "steady." The *vis viva* is constant throughout the steady motion, and the same oscillations follow from the same disturbance at whatever instant it may be applied to the system.

We must now substitute the values of T and U in Lagrange's equations

$$\frac{d}{dt}\frac{d\mathrm{T}}{d\theta'} - \frac{d\mathrm{T}}{d\theta} = \frac{d\mathrm{U}}{d\theta},$$

$$\&c. = \&c.,$$

and reject the squares of small quantities. We thus get as many equations as there are independent variables. The steady motion being given by θ, ϕ, &c. all zero, each of these must be satisfied when we omit the terms containing θ, ϕ, &c. Thus we get the equations of steady motion. Omitting these, the equations may be written in the forms—

θ	ϕ	ψ	&c.
$T_{\theta'\theta'}D^2 - T_{\theta\theta} - U_{\theta\theta}$	$T_{\theta'\phi'}D^2 - T_{\theta\phi} - U_{\theta\phi} + (T_{\theta'\phi} - T_{\theta\phi'})D$	$T_{\theta'\psi'}D^2 - T_{\theta\phi} - U_{\theta\phi} + (T_{\theta'\psi} - T_{\theta\psi'})D$	&c.
$T_{\theta'\phi'}D^2 - T_{\theta\phi} - U_{\theta\phi} + (T_{\theta\phi'} - T_{\theta'\phi})D$	$T_{\phi'\phi'}D^2 - T_{\phi\phi} - U_{\phi\phi}$	$T_{\phi'\psi'}D^2 - T_{\phi\psi} - U_{\phi\psi} + (T_{\phi'\psi} - T_{\phi\psi'})D$	&c.
$T_{\theta'\psi'}D^2 - T_{\theta\psi} - U_{\theta\psi} + (T_{\theta\psi'} - T_{\theta\psi})D$	$T_{\psi'\phi'} - T_{\psi\phi} - U_{\psi\phi} + (T_{\psi'\phi} - T_{\psi\phi'})D$	$T_{\psi\psi'}D^2 - T_{\psi\psi} - U_{\psi\psi}$	&c.
&c.	&c.	&c.	&c.

$= 0.$

The operators on θ, ϕ, ψ, &c. forming the successive series of the determinant, and D standing for $\frac{d}{dt}$.

If we put $\theta = Ae^{\lambda t}$, &c., this determinant, with λ written for D, is the equation to find λ. If we change the sign of D, the columns become rows, and the rows columns; thus the determinant is unchanged, and therefore the equation has no *odd* powers of D. We have, therefore, an equation to find D^2. If the roots be all real and negative, the

motion is stable, and any simple oscillation is given by

$$\theta = A_1 \sin \mu t + B_1 \cos \mu t,$$
$$\phi = A_2 \sin \mu t + B_2 \cos \mu t, \quad \&c.$$

If any one of the roots be imaginary, or real and positive, we shall have both positive and negative real exponentials entering into the expressions for θ, ϕ, &c.; and therefore the motion will be unstable.

The chief labour in any application will be the expansion of this determinant, and the subsequent solution of the equation. Whenever, therefore, we have more than three independent variables, we should try to choose them so that some at least of the roots may be apparent without expanding the determinant.

If T and U are independent of any coordinate, say x, whose corresponding mean motion nt does not vanish, though not necessarily independent of the differential coefficients of x, we have

$$T_{\theta\theta} = 0,\ T_{\theta\phi} = 0,\ \&c.,\ U_{\theta\theta} = 0,\ U_{\theta\phi} = 0,\ \&c.$$

Hence the constant terms in every constituent in the first line must vanish. We may then divide out the D from the first line and the first column, thus reducing the determinant two orders. The same argument applies to each coordinate containing nt. Thus there will be one periodic term less for every combination of coordinates which can be made constant in the steady motion, though the remaining periodic terms may occur in all the coordinates. This remark seems important, as it is often useful to know beforehand a superior limit to the number of periods of oscillation the system admits of. The zero roots may be omitted, for their effect is only to alter the mean motions of the coordinates, and these may be regarded as arbitrary in the first instance. The determinant being freed from these roots, its expansion may be simplified by methods which it is unnecessary to dwell on here.

Next suppose we know beforehand the nature of one of the harmonic oscillations, and that we can choose the coordinates so that it is produced by a variation of only one coordinate, say θ, then the equations formed by equating the first column to zero may all be satisfied by $\theta = A \sin \mu t + B \cos \mu t$. Hence the coefficients of D in that column must all be zero. The factor $\lambda^2 + \mu^2$ may then be brought out of the determinant.

So, also, if a possible oscillation can be represented by a variation of only two coordinates, say θ, ϕ; then, by adding to the first column the second column multiplied by some constant, we may again divide out by a quadratic factor, and reduce the determinant two orders.

These remarks are suggested and illustrated by the problem considered of the three particles. For, though the determinant was not deduced from Lagrange's equations, yet, however obtained, the resulting equation must be the same.

Note on some Relations between certain Elliptic and Hyperbolic Functions. By John Griffiths, M.A.

[*Read February 11th*, 1875.]

If the definite integrals

$$\int_{\theta_0}^{\theta} \sqrt{1-e^2\sin^2\theta}\,d\theta, \quad \int_{\theta_0}^{\theta} \frac{d\theta}{\sqrt{1-e^2\sin^2\theta}}, \text{ and } \int_{\theta_0}^{\theta} \sqrt{e'^2\operatorname{cosec}^2\theta-1}\,\operatorname{cosec}\theta\,d\theta$$

be denoted by $\mathrm{E}^{\theta}_{\theta_0}$, $\mathrm{F}^{\theta}_{\theta_0}$, and $\mathrm{H}^{\theta}_{\theta_0}$ respectively, where $ee'=1$, I propose to show, in the first place, that they are connected by the equation

$$e\mathrm{H}^{\theta}_{\theta_0}+\mathrm{E}^{\theta}_{\theta_0}-(1-e^2)\,\mathrm{F}^{\theta}_{\theta_0}+\left[\sqrt{1-e^2\sin^2\theta}\,\cot\theta\right]^{\theta}_{\theta_0}=0 \quad \ldots\ldots (1);$$

and also, if

$$\cos\theta\cos\phi-\sin\theta\sin\phi\sqrt{1-e^2\sin^2\mu}$$
$$=\cos\mu=\cos\theta_0\cos\phi_0-\sin\theta_0\sin\phi_0\sqrt{1-e^2\sin^2\mu},$$

(where μ is a constant), to deduce from (1) the further relation

$$\frac{\mathrm{E}^{\theta}_{\theta_0}+\mathrm{E}^{\phi}_{\phi_0}}{\mathrm{H}^{\theta}_{\theta_0}+\mathrm{H}^{\phi}_{\phi_0}}=e^3\sin\theta\sin\phi\sin\theta_0\sin\phi_0 \quad \ldots\ldots\ldots\ldots (2).$$

[The integral $\int_{\theta_0}^{\theta}\sqrt{1-e^2\sin^2\theta}\,d\theta$ expresses, as we know, the length of the arc of the ellipse $x^2+\frac{y^2}{1-e^2}=1$ between the points P, P_0, whose abscissæ are $x=\sin\theta$, $x_0=\sin\theta_0$ respectively; and if $e'=\frac{1}{e}$, the integral $\int_{\theta_0}^{\theta}\sqrt{e'^2\operatorname{cosec}^2\theta-1}\,\operatorname{cosec}\theta\,d\theta$ denotes the length of the arc of the hyperbola $x^2-\frac{e^2y^2}{1-e^2}=1$ between the corresponding points P′, P'_0, whose abscissæ are $x=\operatorname{cosec}\theta$, $x_0=\operatorname{cosec}\theta_0$.]

Taking, then, $\mathrm{H}^{\theta}_{\theta_0}=\int_{\theta_0}^{\theta}\sqrt{e'^2\operatorname{cosec}^2\theta-1}\,\operatorname{cosec}\theta\,d\theta$, and $e'=\frac{1}{e}$, we have

$$-e\mathrm{H}^{\theta}_{\theta_0}=-\int_{\theta_0}^{\theta}\frac{\sqrt{1-e^2\sin^2\theta}}{\sin^2\theta}\,d\theta$$
$$=\left[\sqrt{1-e^2\sin^2\theta}\,\cot\theta\right]^{\theta}_{\theta_0}+\int_{\theta_0}^{\theta}\frac{e^2\cos^2\theta}{\sqrt{1-e^2\sin^2\theta}}\,d\theta$$
$$=\left[\sqrt{1-e^2\sin^2\theta}\,\cot\theta\right]^{\theta}_{\theta_0}+\int_{\theta_0}^{\theta}\sqrt{1-e^2\sin^2\theta}\,d\theta-(1-e^2)\int_{\theta_0}^{\theta}\frac{d\theta}{\sqrt{1-e^2\sin^2\theta}};$$

i. e., $$eH^{\theta}_{\theta_0}+E^{\theta}_{\theta_0}-(1-e^2)\,F^{\theta}_{\theta_0}+\left[\sqrt{1-e^2\sin^2\theta}\,\cot\theta\right]^{\theta}_{\theta_0}=0,$$

which is relation (1).

If θ, θ_0 be connected with two other amplitudes ϕ and ϕ_0, by the relations $\cos\theta\cos\phi-\sin\theta\sin\phi\sqrt{1-e^2\sin^2\mu}=\cos\mu$

$$=\cos\theta_0\cos\phi_0-\sin\theta_0\sin\phi_0\sqrt{1-e^2\sin^2\mu};$$

or, what is the same thing, by

$$\left.\begin{aligned}\cos\theta&=\cos\phi\cos\mu+\sin\phi\sin\mu\sqrt{1-e^2\sin^2\theta}\\ \cos\theta_0&=\cos\phi_0\cos\mu+\sin\phi_0\sin\mu\sqrt{1-e^2\sin^2\theta_0}\end{aligned}\right\}\quad\ldots\ldots\ldots(3),$$

the equation (1) becomes

$$eH^{\theta}_{\theta_0}+E^{\theta}_{\theta_0}-(1-e^2)\,F^{\theta}_{\theta_0}+\left[\frac{\cos^2\theta-\cos\theta\cos\phi\cos\mu}{\sin\theta\sin\phi\sin\mu}\right]^{\theta}_{\theta_0}=0.$$

In the same way, we have

$$eH^{\phi}_{\phi_0}+E^{\phi}_{\phi_0}-(1-e^2)\,F^{\phi}_{\phi_0}+\left[\frac{\cos^2\phi-\cos\theta\cos\phi\cos\mu}{\sin\theta\sin\varphi\sin\mu}\right]^{\phi}_{\phi_0}=0;$$

hence, by addition,

$$\begin{aligned}e\,(H^{\theta}_{\theta_0}+H^{\phi}_{\phi_0})+(E^{\theta}_{\theta_0}+E^{\phi}_{\phi_0})-(1-e^2)\,(F^{\theta}_{\theta_0}+F^{\phi}_{\phi_0})&\\ +\frac{\cos^2\theta+\cos^2\phi-2\cos\theta\cos\phi\cos\mu}{\sin\theta\sin\varphi\sin\mu}&\\ -\frac{\cos^2\theta_0+\cos^2\phi_0-2\cos\theta_0\cos\phi_0\cos\mu}{\sin\theta_0\sin\phi_0\sin\mu}&=0;\end{aligned}$$

or, since (3) are equivalent to the following, viz.,

$$\cos^2\theta+\cos^2\phi-2\cos\theta\cos\phi\cos\mu=\sin^2\mu\,(1-e^2\sin^2\theta\sin^2\phi),$$
$$\cos^2\theta_0+\cos^2\phi_0-2\cos\theta_0\cos\phi_0\cos\mu=\sin^2\mu\,(1-e^2\sin^2\theta_0\sin^2\phi_0),$$

$$\begin{aligned}e\,(H^{\theta}_{\theta_0}+H^{\phi}_{\phi_0})+(E^{\theta}_{\theta_0}+E^{\phi}_{\phi_0})-(1-e^2)\,(F^{\theta}_{\theta_0}+F^{\phi}_{\phi_0})&\\ +\sin\mu\,(\operatorname{cosec}\theta\ \operatorname{cosec}\varphi-e^2\sin\theta\ \sin\phi)&\\ -\sin\mu\,(\operatorname{cosec}\theta_0\operatorname{cosec}\phi_0-e^2\sin\theta_0\sin\phi_0)&=0;\end{aligned}$$

i. e., as we know that $E^{\theta}_{\theta_0}+E^{\phi}_{\phi_0}=e^2\sin\mu\,(\sin\theta\sin\phi-\sin\theta_0\sin\phi_0)$, and $F^{\theta}_{\theta_0}+F^{\phi}_{\phi_0}=0$, it follows that

$$e\,(H^{\theta}_{\theta_0}+H^{\phi}_{\phi_0})+\sin\mu\,(\operatorname{cosec}\theta\operatorname{cosec}\phi-\operatorname{cosec}\theta_0\operatorname{cosec}\phi_0)=0;$$

and consequently $$\frac{E^{\theta}_{\theta_0}+E^{\phi}_{\phi_0}}{H^{\theta}_{\theta_0}+H^{\phi}_{\phi_0}}=e^3\sin\theta\sin\phi\sin\theta_0\sin\phi_0,$$

which is relation (2).

I may remark, that the locus of the point of intersection of the

tangents to the ellipse $x^2 + \frac{y^2}{1-e^2} = 1$, at the points θ and ϕ, is a hyperbola confocal with the ellipse.

This may be proved as follows: —

The equations of the tangents in question are

$$X \sin\theta + \frac{Y}{1-e^2}\cos\theta = 1,$$

$$X \sin\phi + \frac{Y}{1-e^2}\cos\phi = 1,$$

or, say, $X' \sin\theta + Y' \cos\theta = 1$, and $X' \sin\phi + Y' \cos\phi = 1$,

where $X' = X$, and $Y' = \frac{Y}{\sqrt{1-e^2}}$;

i. e., $\sin\theta$, $\sin\phi$ and $\cos\theta$, $\cos\phi$ are the roots of the two respective quadratics

$$(X'^2+Y'^2)\sin^2\theta - 2X'\sin\theta + 1 - Y'^2 = 0,$$
$$(X'^2+Y'^2)\cos^2\theta - 2Y'\cos\theta + 1 - X'^2 = 0.$$

Hence, since $\cos\theta\cos\phi - \sin\theta\sin\phi\sqrt{1-e^2\sin^2\mu} = \cos\mu$,

$$\frac{1-X'^2}{X'^2+Y'^2} - \frac{1-Y'^2}{X'^2+Y'^2}\sqrt{1-e^2\sin^2\mu} = \cos\mu,$$

which, on our replacing X', Y' by X, $\frac{Y}{\sqrt{1-e^2}}$, reduces to the following, viz., $\frac{X^2}{1-\cos\mu} - \frac{Y^2}{\cos\mu + \sqrt{1-e^2\sin^2\mu}} = \frac{e^2}{1+\sqrt{1-e^2\sin^2\mu}}$; a hyperbola confocal with the ellipse $x^2 + \frac{y^2}{1-e^2} = 1$.

In the same way it may be shown that the locus of the intersection of tangents to the hyperbola $x^2 - \frac{e^2y^2}{1-e^2} = 1$, at the points P', Q', ($x = \operatorname{cosec}\theta$ and $x = \operatorname{cosec}\phi$) is an ellipse confocal with the hyperbola.

[Compare Mr. MacCullagh's theorem, viz.: If two tangents be drawn to an ellipse from any point of a confocal hyperbola, the difference of the arcs PK, QK is equal to the difference of the tangents TP, TQ. —See Dr. Salmon's "Conic Sections," p. 358, 5th ed.]

March 11*th*, 1875.

Prof. H. J. S. SMITH, F.R.S., President, in the Chair.

Messrs. A. B. Kempe, B.A., and S. A. Renshaw were elected Members; Mr. J. H. Röbrs, M.A., was proposed for election; and Mr. Harry Hart, M.A., was admitted into the Society.

Mr. Roberts gave an account of his paper, "On a Simplified Method of obtaining the Order of Algebraical Conditions."

Mr. Sylvester, F.R.S., spoke on the subject of "An Orthogonal Web," pointing out a curious paradox when the reticulation was not all in the same plane.

Mr. Tucker read a portion of Mr. Darwin's paper, "On some proposed forms of Slide-rule."

The following presents were received:—

"Bulletin de la Société Mathématique de France," Tome ii., Fev. No. 5.

"Mémoires de la Société des Sciences Physiques et Naturelles de Bordeaux," Tome i., 2e Série, 1er Cahier. Paris, 1875.

"Bemerkungen zur Theorie der Ternären cubichen Formen von Axel Harnack." Erlangen, vom 8 Febr., 1875.

"Jahrbuch über die Fortschritte der Mathematik," viertes Band, Jahrgang 1872, Heft 3.

"Journal of the Institute of Actuaries," No. 97, Oct. 1874.

"Fifth Annual Report of the Association for the Improvement of Geometrical Teaching," January, 1875.

"Table des Fonctions Symétriques de Poids XI," dressée par le Chev. F. Faà de Bruno (extrait de la Théorie des Formes Binaires, du même auteur), Mars, 1875.

On a Simplified Method of obtaining the Order of Algebraical Conditions. By S. Roberts, M.A.

[*Read March* 11*th*, 1875.]

1. I propose to give some examples of a method of obtaining the order of the conditions for the co-existence of systems of equations. The method easily leads to the required expressions in the simpler cases, and shows the course of procedure where the actual expression is complicated and need not be evaluated.

One of the simplest examples will be given in detail. A few others have been chosen with a view to determining the characteristics of certain envelope surfaces to which the results will be applied.

2. Let it be required to determine the order of the conditions, two in number, that a pair of binary equations $U=0$, $V=0$, containing homogeneously the parameters α, β, may have two common solutions. The coefficient of the highest power of α (say α^l) in the first equation is supposed to be of the order λ in uneliminated variables; that of $\alpha^{l-1}\beta$ is of the order $\lambda+a$; that of $\alpha^{l-2}\beta^2$ is of the order $\lambda+2a$, and so on. In the second equation, the coefficient of the highest power of α (say α^m) is of the order μ; that of $\alpha^{m-1}\beta$ is of the order $\mu+a$; that of $\alpha^{m-2}\beta^2$ is of the order $\mu+2a$, and so on. In other words, the equations are of the forms

$$U = \Sigma A_{\lambda+pa}\, \alpha^{l-p}\beta^p, \qquad p \not> l,$$
$$V = \Sigma B_{\mu+pa}\, \alpha^{m-p}\beta^p, \qquad p \not> m,$$

the suffixes denoting the order in which the uneliminated variables enter the coefficients.

Writing N_m for the order required, so that N_{m+1} denotes the corresponding order when m is changed into $m+1$, we have first of all to determine $N_{m+1}-N_m$ or ΔN_m. This difference being integrated, the result is to be made symmetrical with regard to l and m, λ and μ.

There will generally remain a constant to be determined by special values of l and m. This is the general process which will be employed in all cases.

I use the notation $\left|\begin{matrix}U\\V\\\&c.\end{matrix}\right|_i$ to denote the order of the conditions that $U=0$, $V=0$, &c., may have i common solutions.

In the case before us, suppose that V is replaced by V . X, where X is a linear factor $A\alpha+B\beta$. The coefficients A and B may be taken to be of the orders zero and a respectively with regard to the uneliminated variables, so that in effect the orders of the coefficients of V . X remain the same as those of V, while the degree of the equation is increased by unity. But we may, where it becomes expedient, simplify still further; for X may be considered as identical with α, one of the parameters, and the factor is then of the order zero as to the uneliminated variables. The effect of the breaking up of the equation into V and a linear factor must be taken into consideration and allowed for. Thus we have

$$N_{m+1} = N_m + {}_2\left|\begin{matrix}U\\X\end{matrix}\right| + {}_1\left|\begin{matrix}U\\V\end{matrix}\right|_1\left|\begin{matrix}U\\X\end{matrix}\right| - {}_1\left|\begin{matrix}U\\V\\X\end{matrix}\right|$$

That is to say, the conditions of the case are fulfilled if U and V have two common solutions, or if U and X have two common solutions,

or if U and V have a common solution, and U and X have another different common solution. We must consequently deduct the order of the conditions that U, V, and X may have a common solution.

The order $\begin{vmatrix} U \\ X \end{vmatrix}_2$ is zero, so that

$$\Delta N_m = \begin{vmatrix} U \\ V \end{vmatrix}_1 \begin{vmatrix} U \\ X \end{vmatrix} - \begin{vmatrix} U \\ V \\ X \end{vmatrix}_1 = (\lambda m + \mu l + lma)(\lambda + la) - (\lambda + la)(\mu + ma)$$

$$\text{and } N_m = \lambda^2 \frac{m^2 - m}{2} + \lambda\mu (l-1) m + \lambda a (2l-1) \frac{m^2-m}{2}$$

$$+ \mu a (l^2 - l) m + a^2 (l^2 - l) \frac{m^2-m}{2} + F.$$

To make this symmetrical in l and m, λ and μ, we must take

$$F = \mu^2 \frac{l^2-l}{2} - \lambda\mu l - \mu a \frac{l^2-l}{2} + C.$$

The expression must vanish for $l=1$, $m=1$, so that $C = \lambda\mu$ and the result is

$$\lambda^2 \frac{m^2-m}{2} + \mu^2 \frac{l^2-l}{2} + \lambda\mu (l-1)(m-1)$$

$$+ \frac{\lambda a}{2} m (m-1)(2l-1) + \frac{\mu a}{2} l(l-1)(2m-1) + \frac{a^2}{2} lm (l-1)(m-1),$$

and agrees with that arrived at in other ways.

3. To obtain the order of the conditions that two binary equations may have two common solutions, one of them a solution of a third equation, we may proceed in the same way.

Let the third equation R=0 be of the degree n, the coefficient being of the order γ in uneliminated variables. For simplicity's sake we will here take $a=0$.

Then, if $\begin{vmatrix} R \\ U \end{vmatrix}_1$ denote the order of the conditions that U, V may have two common solutions, one of them a solution of R,

$$N_{m+1} = \begin{vmatrix} R \\ U \\ V.X \end{vmatrix}_2 = \begin{vmatrix} R \\ U \\ V \end{vmatrix}_2 + \begin{vmatrix} R \\ U \\ X \end{vmatrix}_2 + \begin{vmatrix} R \\ U \\ V \end{vmatrix}_1 \begin{vmatrix} U \\ X \end{vmatrix}_1 + \begin{vmatrix} R \\ U \\ X \end{vmatrix}_1 \begin{vmatrix} U \\ V \end{vmatrix}_1 - 2 \begin{vmatrix} R \\ U \\ V \\ X \end{vmatrix}_1$$

But $\begin{vmatrix} R \\ U \\ X \end{vmatrix}_2 = 0$, and therefore

$$\Delta N_m = \begin{vmatrix} R \\ U \\ V \end{vmatrix}_1 \begin{vmatrix} U \\ X \end{vmatrix}_1 + \begin{vmatrix} R \\ U \\ X \end{vmatrix}_1 \begin{vmatrix} U \\ V \end{vmatrix}_1 - 2 \begin{vmatrix} R \\ U \\ V \\ X \end{vmatrix}_1$$

$$= \lambda (\mu\nu l + \lambda\nu m + \lambda\mu n) + \lambda\nu (\lambda m + \mu l) - 2\lambda\mu\nu,$$

$$\text{and} \qquad N_m = \lambda^2\nu (m^2 - m) + [\lambda^2\mu n + 2\lambda\mu\nu l - 2\lambda\mu\nu] m + F;$$

and we must write for F,

$$\mu^2\nu\,(l^2-l) + \mu^2\lambda l n - 2\lambda\mu\nu l + C.$$

The expression must vanish for $l=1$, $m=1$, giving

$$C + (\lambda^2\mu + \mu^2\lambda)\,n - 2\lambda\mu\nu = 0.$$

Finally,

$$N_m = \nu\,\{\lambda^2(m^2-m) + \mu^2(l^2-l) + 2\lambda\mu\,(l-1)\,(m-1)\} + n\lambda\mu\,\{\lambda(m-1) + \mu(l-1)\}.$$

4. Again, let it be required to find the order of the conditions that three ternary equations may have two common solutions, one of them a solution of a fourth equation.

The formation of the equations is supposed to be analogous to that of the previous binary equations, viz., they are of the form

$$\Sigma\, A_{\lambda+p\alpha+q\beta}\,\alpha^{l-p-q}\,\beta^p\,\gamma^q, \qquad p+q \nleq l.$$

We have, using an obvious extension of the notation,

$$N_n = {}_1\left|\begin{matrix}R\\U\\V\\W.X\end{matrix}\right|_2 = {}_1\left|\begin{matrix}R\\U\\V\\W\end{matrix}\right|_2 + {}_1\left|\begin{matrix}R\\U\\V\\X\end{matrix}\right|_2 + \left|\begin{matrix}R\\U\\V\\W\end{matrix}\right|_1\left|\begin{matrix}U\\V\\X\end{matrix}\right|_1 + \left|\begin{matrix}R\\U\\V\\X\end{matrix}\right|_1\left|\begin{matrix}U\\V\\W\end{matrix}\right|_1 - 2\left|\begin{matrix}R\\U\\V\\W\\X\end{matrix}\right|_1$$

or

$$\Delta N_n = {}_1\left|\begin{matrix}R\\U\\V\\X\end{matrix}\right|_2 + \left|\begin{matrix}R\\U\\V\\W\end{matrix}\right|_1\left|\begin{matrix}U\\V\\X\end{matrix}\right|_1 + \left|\begin{matrix}R\\U\\V\\X\end{matrix}\right|_1\left|\begin{matrix}U\\V\\W\end{matrix}\right|_1 - 2\left|\begin{matrix}R\\U\\V\\W\\X\end{matrix}\right|_1$$

${}_1\left|\begin{matrix}R\\U\\V\\X\end{matrix}\right|_2$ is known, since we have only to put a for X, and then make $a=0$, when the form is reduced to ${}_1\left|\begin{matrix}(R)\\(U)\\(V)\end{matrix}\right|_2$ where (R), (U), (V) are binary.

It is unnecessary here to work out the general formula. We shall only require the case in which the coefficients of R are constant, and $a=0$, $b=0$. The degrees of R, U, V, W being p, l, m, n, and the orders of the coefficients 0, λ, μ, ν, we have

$$\Delta N_n = p\left\{\begin{matrix}\lambda\mu\,\{\lambda(m-1) + \mu(l-1)\} + (\lambda m+\mu l)(\mu\nu l+\lambda\nu m+\lambda\mu n)\\ + \lambda\mu\,(\lambda mn+\mu ln+\nu lm) - 2\lambda\mu\nu\end{matrix}\right\},$$

and

$$N_n = p\left\{\begin{matrix}\lambda\mu\,(\lambda m+\mu l)\,(n^2-n) + \left\{\begin{matrix}\lambda\mu\,[\lambda(m-1)+\mu(l-1)]\\ +(\lambda m+\mu l)\,(\mu\nu l+\lambda\nu m)\\ +\lambda\mu\nu lm-2\lambda\mu\nu\end{matrix}\right\}n\\ + F.\end{matrix}\right\}$$

Making this symmetrical, and taking the constant so that the expression may vanish for $l=m=n=1$, we get

$$N_n = p\{\Sigma\lambda^2\mu n\,(mn-1) + \lambda\mu\nu\,(3lmn-2\Sigma l+3)\}.$$

Generally, for $k+1$ equations in all, we have

$$\Delta N = {}_1\begin{vmatrix} R \\ U_1 \\ U_2 \\ \vdots \\ U_{k-1} \\ {}_2\,X \end{vmatrix} + {}_1\begin{vmatrix} R \\ U_1 \\ U_2 \\ \vdots \\ {}_1\,U_k \end{vmatrix} \begin{vmatrix} U_1 \\ U_2 \\ \vdots \\ U_{k-1} \\ X \end{vmatrix} + {}_1\begin{vmatrix} R \\ U_1 \\ U_2 \\ \vdots \\ {}_1\,U_{k-1} \\ X \end{vmatrix} \begin{vmatrix} U_1 \\ U_2 \\ \vdots \\ U_k \end{vmatrix} - 2 {}_1\begin{vmatrix} R \\ U_1 \\ U_2 \\ \vdots \\ U_k \\ X \end{vmatrix}$$

the case of $k+1$ equations is thus reduced to that of k equations and other known forms.

5. To find the order of the conditions that three ternary equations may have a pair of coincident solutions, or, what is the same thing, that three curves may touch at the same point. This is also a case which will be required subsequently. Using t to denote the condition of touching, we have

$$N_{n+1} = {}_t\begin{vmatrix} U \\ V \\ W \end{vmatrix} + {}_t\begin{vmatrix} U \\ V \\ X \end{vmatrix} + 2\,{}^t{}_1\begin{vmatrix} U \\ V \\ W \\ X \end{vmatrix}$$

that is to say, we may have U, V, W touching at the same point, or U, V having a common tangent X, with the same point of contact; or we may have U, V touching at an intersection of W and X. This, we know, counts as two contacts.

Finally, then,

$$\Delta N_n = \begin{vmatrix} U \\ V \\ X \end{vmatrix} + 2\,{}_1\begin{vmatrix} U \\ V \\ W \\ X \end{vmatrix} = 2\lambda\mu\{\lambda(\mu-1)+\mu(l-1)\}+2\,{}_2\begin{vmatrix} U \\ V \\ W \\ X \end{vmatrix}$$

We will suppose here also that $a=0$, $b=0$, a and b being the increments of the orders of the coefficients, as we proceed from term to term. But

$${}^t{}_1\begin{vmatrix} U \\ V \\ W \\ X \end{vmatrix} = {}_1\begin{vmatrix} U_1V_2-U_2V_1 \\ U \\ V \\ W \\ X \end{vmatrix} - {}_1\begin{vmatrix} U_1V_2-U_2V_1 \\ V \\ \gamma \\ W \\ X \end{vmatrix} + {}_1\begin{vmatrix} V_1 \\ V_2 \\ \gamma \\ W \\ X \end{vmatrix}$$

where U_1 is written for $\frac{dU}{da}$, U_2 for $\frac{dU}{d\beta}$ &c., and γ is the third variable.

Consequently,

$$\begin{matrix} t \\ \\ 1 \end{matrix}\left|\begin{matrix} U \\ V \\ W \\ X \end{matrix}\right| = (\lambda+\mu)(\lambda\mu n+\lambda\nu m+\mu\nu l)+\lambda\mu\nu(l+m-2)-(\lambda+\mu)\mu\nu+\mu^2\nu$$

and

$$\Delta N_n = 2\lambda^2\mu(m+n-1)+2\lambda\mu^2(l+n-1)+2\lambda^2\nu m+2\mu^2\nu l +2\lambda\mu\nu(2l+2m-3).$$

Hence,

$$N_n = \lambda^2\mu(n^2-3n+2mn)+\lambda\mu^2(n^2-3n+2ln) +2\lambda^2\nu mn+2\mu^2\nu ln+2\lambda\mu\nu(2l+2m-3)n+F.$$

The expression must vanish for $l=m=n=1$, so that ultimately

$$N_n = \Sigma\lambda^2\mu(n^2-3n+2mn)+2\lambda\mu\nu\{2\Sigma lm-3\Sigma l+3\}.$$

6. I might proceed in the same manner to determine the order of the conditions that two binary equations may have three common solutions. The formula is, however, well known, and I only write it down for the sake of convenience, as it will be required presently. The order of the coefficients of the two equations being λ, μ respectively, and their degrees being l, m, we have (Salmon's "Lessons on Higher Algebra," p. 229)

$$\frac{m(m-1)(m-2)}{2.3}\lambda^3+\frac{l(l-1)(l-2)}{2.3}\mu^3 +\tfrac{1}{2}(m-1)(m-2)(l-2)\lambda^2\mu+\tfrac{1}{2}(l-1)(l-2)(m-2)\lambda\mu^2.$$

To obtain the order of the conditions that three ternary equations may have three common solutions, we set out with the following equality:—

$$N_{n+1} = \begin{matrix} \\ 3 \end{matrix}\left|\begin{matrix} U \\ V \\ W.X \end{matrix}\right| = \begin{matrix} \\ 3 \end{matrix}\left|\begin{matrix} U \\ V \\ W \end{matrix}\right| + \begin{matrix} \\ 3 \end{matrix}\left|\begin{matrix} U \\ V \\ X \end{matrix}\right| + \begin{matrix} \\ 2 \end{matrix}\left|\begin{matrix} U \\ V \\ W \end{matrix}\right|\begin{matrix} U \\ \Lambda \\ X \end{matrix}\Big| + \begin{matrix} \\ 1 \end{matrix}\left|\begin{matrix} U \\ V \\ W_2 \end{matrix}\right|\begin{matrix} U \\ V \\ X \end{matrix}\Big| - \begin{matrix} 1 \\ \\ 2 \end{matrix}\left|\begin{matrix} W \\ U \\ V \\ X \end{matrix}\right| - \begin{matrix} 1 \\ \\ 2 \end{matrix}\left|\begin{matrix} X \\ U \\ V \\ W \end{matrix}\right|$$

or

$$\Delta N_n = \begin{matrix} \\ 3 \end{matrix}\left|\begin{matrix} U \\ \Lambda \\ X \end{matrix}\right| + \begin{matrix} \\ 2 \end{matrix}\left|\begin{matrix} U \\ V \\ W \end{matrix}\right|\begin{matrix} U \\ V \\ X \end{matrix}\Big| + \begin{matrix} \\ 2 \end{matrix}\left|\begin{matrix} U \\ V \\ X \end{matrix}\right|\begin{matrix} U \\ V \\ W \end{matrix}\Big| - \begin{matrix} 1 \\ \\ 2 \end{matrix}\left|\begin{matrix} W \\ U \\ V \\ X \end{matrix}\right| - \begin{matrix} 1 \\ \\ 2 \end{matrix}\left|\begin{matrix} X \\ U \\ V \\ W \end{matrix}\right|$$

I shall only require the case in which the coefficients of each equation are of the same order throughout.

We have for this case, using the preceding formula,

$$\Delta N_n = \frac{ln(m-1)(m-2)}{2.3}\lambda^3+\frac{l(l-1)(l-2)}{2.3}\mu^3+\tfrac{1}{2}l(l-1)(m-2)\lambda\mu^2 +\tfrac{1}{2}m(m-1)(l-2)\lambda^2\mu$$

$$+(\lambda m+\mu l)\left\{\Sigma\frac{lm\,(lm-1)}{2}\nu^2+\Sigma\{(ln-1)(lm-1)-\tfrac{1}{2}(l-1)(l-2)\}\mu\nu\right\}$$
$$+(\lambda mn+\mu ln+\nu lm)\left(\frac{l(l-1)}{2}\mu^2+\frac{m(m-1)}{2}\lambda^2+(l-1)(m-1)\lambda\mu\right)$$
$$-\{\Sigma\mu\nu l(ln-1)+(3lmn-2\Sigma l+3)\lambda\mu\nu\}$$
$$-\{\nu[l^2-l)\mu^2+(m^2-m)\lambda^2+2(l-1)(m-1)\lambda\mu]$$
$$+n\lambda\mu[\mu(l-1)+\lambda(m-1)]\}.$$

The expression for N_n must vanish for $l=m=n=1$, and must be made symmetrical.

Thus, for the coefficient of λ^3 in N_n, we have to integrate

$$\frac{m(m-1)(m-2)}{2.3}+\frac{m^3n^2-m^2n}{2}+\frac{m^3n-m^2n}{2},$$

and we get

$$\tfrac{1}{2}\left\{m^3\left(\frac{n^3}{3}-\frac{n^2}{2}+\frac{n}{2.3}\right)+(m^3-2m^2)\frac{n^2-n}{2}+\frac{m(m-1)(m-2)}{3}n\right\}$$
$$=\frac{1}{2.3}mn\,(mn-1)(mn-2).$$

In the same way the coefficients of $\lambda^2\mu$, &c. are obtained, and I find

$$N_n=\Sigma\frac{mn\,(mn-1)(mn-2)}{2.3}\lambda^3$$
$$+\Sigma\mu^2\nu\{(lm-2)(ln-1)(ln-2)-(ln-2)(l-1)(l-2)\}$$
$$+\frac{\lambda\mu\nu}{2}\left\{2l^2m^2n^2-5lmn\,\Sigma l+2\Sigma l^2+8\Sigma lm-12\Sigma l+9lmn+10\right\}.$$

In a similar manner the value of ΔN_n can be reduced to known forms when there are k equations homogeneous in k variables. The form is

$$\Delta N_n=\begin{vmatrix}U_1\\U_2\\\vdots\\U_{k-1}\\X\end{vmatrix}_3+\begin{vmatrix}U_1\\U_2\\\vdots\\U_k\end{vmatrix}_2\begin{vmatrix}U\\U_2\\\vdots\\U_{k-1}\\X\end{vmatrix}_1+\begin{vmatrix}U_1\\U_2\\\vdots\\U_{k-1}\\X\end{vmatrix}_2\begin{vmatrix}U_1\\U_2\\\vdots\\U_k\end{vmatrix}_1-\begin{vmatrix}U_k\\U_1\\\vdots\\U_{k-1}\\X\end{vmatrix}_2-\begin{vmatrix}X\\U_1\\U_2\\\vdots\\U_k\end{vmatrix}_2$$

The determination of the order is thus reduced to the case of $k-1$ equations in $k-1$ variables and other lower forms.

My intention here is to indicate a method, and show how it may prove useful. I have, therefore, only taken a few cases which are applicable to the geometrical problems I now proceed to consider.

Geometrical Applications.

7. In plane space the ordinary singularities of a curve ultimately depend on two independent conditions in the coordinates. For two

such conditions determine punctually or tangentially a definite number of singular elements. On the other hand, in space of three dimensions we have to take into consideration systems of three independent conditions which determine groups of singular elements, points, or planes, as well as systems of two conditions determining singular curves on the surface, or developables generated by singular tangent planes.

Thus, if we have two equations,

$$\left.\begin{aligned} \mathrm{U} &= a\,t^m + b\,t^{m-1} + c\,t^{m-2} + \&c. = 0 \\ \mathrm{V} &= a'\,t^{m'} + b'\,t^{m'-1} + c'\,t^{m'-2} + \&c. = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ (\mathrm{A}),$$

a, b, &c. being homogeneous functions of the order μ in the plane coordinates x, y, z, and a', b', &c. being similar functions of the order μ', the resultant with regard to t equated to zero represents a curve of the order $\mu m' + \mu' m$.

The conditions in x, y, z that the system may have two common solutions give the number of double points on the curve, viz.:—

$$\frac{m^2-m}{2}\mu'^2 + \frac{m'^2-m'}{2}\mu^2 + (m-1)(m'-1)\,\mu\mu' \ \ldots\ldots\ldots\ldots\ (a).$$

But if we consider a, b, &c., a', b', &c., as containing four homogeneous coordinates in the same orders as before, we get, in the same way, a surface whose order is $\mu m' + \mu' m$, and whose double curve is of the order (a).

We must now, however, proceed further, and consider singularities of isolated elements. In the first place, the three conditions that (A) may have three common solutions determine a certain number of triple points on the nodal curve in number

$$\frac{m(m-1)(m-2)}{2\,.\,3}\mu'^3 + \frac{m'(m'-1)(m'-2)}{2\,.\,3}\mu^3$$

$$+\tfrac{1}{2}(m-1)(m-2)(m'-2)\,\mu^2\mu + \tfrac{1}{2}(m'-1)(m'-2)(m-2)\,\mu^2\mu'.$$

In the next place, three conditions are to be fulfilled if (A) have in common pairs of equal roots. The order of these conditions is easily seen to be

$$2\mu\mu'\,[\mu(m'-1) + \mu'(m-1)],$$

and this is the number of cuspidal points of the nodal curve. The remaining singularities can now be determined by means of the known general formulæ.

8. In like manner, if the given equations are three in number, homogeneous in the parameters α, β, γ, and in the coordinates x, y, z, w, and the orders with respect to the parameters are m, m', m'' respectively, and with regard to the coordinates μ, μ', μ'', the order of the surface represented by the resultant equated to zero is $\Sigma\mu m m'$. There is a nodal curve whose order, by the formulæ of my paper on the Plückerian

characteristics, &c. ("Quarterly Journal of Mathematics, Vol. xii., p. 282*) is

$$\tfrac{1}{2}\{\Sigma\mu mm'(\Sigma\mu mm'-\Sigma\mu)-(\Sigma m-3)\,\Sigma\mu\mu'm''\}.$$

Further, we have to consider the three conditions that the given system may have three common solutions. We can at once apply the result of § 6 to this case, and thus have the number of triple points on the surface and the nodal curve.

But, again, we have to consider another case involving three conditions in the coordinates.

It will be observed that the three given equations may have in common two coincident solutions, or, as it may be expressed, the three curves in α, β, γ may have a common point of contact. There are thus a certain number of cuspidal points on the nodal curve, and we have for that number, by § 5,

$$\Sigma\mu^2\mu'(m''^2-3n''+2m'm'')+2\mu\mu'\mu''(2\Sigma mm'-3\Sigma m+3).$$

If there are more than three equations and more than three homogeneous parameters, and the number of equations is equal to the number of parameters, and if the coefficients contain four homogeneous space coordinates, we get, in like manner, a surface represented by the resultant equated to zero. Also, the order of the nodal curve and the number of triple points are determined by the formulæ of the paper before referred to, combined with the formula expressing the order of the conditions that such a system may have three common solutions. I have shown that the expression of this formula is reduced to the difficulty of somewhat lengthy calculations.

When, however, we consider whether there are cuspidal points in general on the nodal curve, it is seen that these will not exist. For the condition to be fulfilled is that there are to be two common coincident solutions of the system, and when the number of equations and parameters is greater than three, more than three conditions in the coordinates must be satisfied.

9. We will now consider the case of envelopes where there is one equation given containing several parameters.

If $U=0$ is the equation containing homogeneously the parameters $a_1, a_2, \ldots a_k$ and the space coordinates x, y, z, w, the envelope we are concerned with is obtained by eliminating the parameters from

$$\frac{dU}{da_1}=0,\quad \frac{dU}{da_2}=0\ \ldots\ldots\ldots\frac{dU}{da_k}=0,$$

and equating the result to zero.

---

* In that paper the applications refer to plane space. The formulæ are, however, applicable, as far as they extend, to space of three dimensions, and I am, therefore, obliged to refer to them several times.

The formulæ of my paper before mentioned (Quarterly Journal, vol. xii., pp. 289, 293, 297) give, therefore, the order of the surface, of its nodal and cuspidal curves. The points j are absorbed in the cuspidal curve. Moreover, when we seek to determine the number of triple points we find included amongst them both the points β, intersections of the nodal and cuspidal curves, stationary points on the cuspidal curve, and the points γ, intersections of the nodal and cuspidal curves stationary points on the nodal curve. That is to say, we get $\beta+\gamma+t$ instead of t, as in the case of simple resultants.

Bearing in mind these considerations, we can now treat the envelope of the planes

$$U = A\alpha^m + B\beta^m + C\gamma^m + \&c. = 0 \ldots\ldots\ldots\ldots\ldots\ldots (B),$$

homogeneous and of the order m in the parameters α, β, γ, and as to the coefficients A, B, C, &c., of the first order in x, y, z, w. This envelope is referred to in Dr. Salmon's "Geometry of Three Dimensions," 3rd ed., p. 537, where he has determined the characteristics in the case of $m=3$.

The order of the surface is the order of the system

$$\frac{dU}{d\alpha}=0, \quad \frac{dU}{d\beta}=0, \quad \frac{dU}{d\gamma}=0 \ldots\ldots\ldots\ldots\ldots\ldots (B'),$$

and is therefore $3(m-1)^2$. The order of the nodal curve is the order of the conditions that the system (B') may have two distinct solutions in common, and is $\frac{(m-1)(m-2)(9m^2-9m-33)}{2}$. Similarly, the order of the cuspidal curve is $12(m-1)(m-2)$.

Again, putting $\lambda=\mu=\nu=1$ and $(m-1)$ for l, m, n in the formula of § 6, we get for $\beta+\gamma+t$

$$\tfrac{1}{2}\{9(m-1)^6-54(m-1)^4+27(m-1)^3+80(m-1)^2-72(m-1)+10\}.$$

For the order of the tangent cone, we condition (B) and

$$U' = A'\alpha^m + B'\beta^m + C'\gamma^m + \&c. = 0$$

(where the accents denote that x, y, z, w are replaced by x', y', z', w') to have two coincident common solutions. The order is $3m(m-1)$. The class of the surface is evidently m^2.

The number of cuspidal edges of the tangent cone is the order of the conditions that $U=0$ and $U'=0$, considered as representing curves in α, β, γ coordinates, may have three-point contact. We have, then, to make $\mu'=0$, $m'=m$, $\epsilon=\eta=0$, in the formulæ given at p. 293 and at p. 297 (Quarterly Journal, vol xii.), by which means we get

$$\delta+\kappa = \tfrac{1}{2}(9m^4-18m^3-m^2+12m),$$
$$\kappa = 6m^2-9m.$$

These values, again, give m^2 for the class. They also show that the reciprocal surface has no cuspidal curve.

We can also determine σ, the number of intersections of the curve of contact of the tangent cone with the cuspidal curve. For this purpose we must combine the equation $U'=0$ with the system

$$\frac{d^2U}{da^2}\cdot\frac{d^2U}{d\beta^2}-\left(\frac{d^2U}{da\,d\beta}\right)^2=0,\ \frac{dU}{da}=0,\ \frac{dU}{d\beta}=0,\ \frac{dU}{d\gamma}=0,$$

and make the same reduction for a double solution satisfying $\gamma=0$ as at p. 285 of the paper before cited. Thus the order sought is the order of a system

$$\begin{vmatrix} 2, & 2m-4 \\ 1, & m-1 \\ 1, & m-1 \\ 1, & m-1 \\ 0, & m \end{vmatrix} \text{ less twice the order of a system } \begin{vmatrix} 0, & 1 \\ 1, & m-1 \\ 1, & m-1 \\ 1, & m-1 \\ 0, & m \end{vmatrix}$$

that is to say,

$$\sigma = 2m\,(m-2)+6m\,(m-1)-2m = 4m\,(2m-3).$$

The value for $\beta+\gamma+t$ is

$$\tfrac{1}{2}\,(9m^6-54m^5+81m^4+63m^3-190m^2+11m+90),$$

and we also have, by the general formulæ for reciprocal surfaces (Salmon's "Geometry of Three Dimensions," p. 543),

$$\begin{aligned} 2\beta+3\gamma+3t &= b\,(m-2)-\rho, \\ 4\beta+\gamma\ \ &= c\,(m-2)-2\sigma. \end{aligned}$$

From this system we get the separate numbers β, γ, t. Also j is found to be zero. The numbers h, k, q, r can now be determined. It remains to consider the tangential singularities. We have already the class, the class of the plane section, and there is no cuspidal curve on the reciprocal. We get, therefore,

$$\kappa' = 3a-3\,(m)+c, \text{ and } 2\delta' = a\,(a-1)-(n)-3\kappa,$$

where (m), (n) are written for the order and class of the surface in the general formulæ. The rest of the required numbers follow in like manner from those formulæ, and the results may be collected as follows:—

Order of surface	$3\,(m-1)^2$
Order of tangent cone	$3m\,(m-1)$
Number of its double edges	$\tfrac{1}{2}\,(9m^4-18m^3-13m^2+30m)$
„ „ cuspidal edges	$3m\,(2m-3)$
Class of nodal torse	$9m^4-27m^3-m^2+30m$
„ cuspidal torse	$4m\,(2m-3)$
Order of nodal curve	$\dfrac{m-1\,.\,m-2}{2}\,(9m^2-9m-33)$

Number of its pinch points 0
Order of cuspidal curve..... $12\,(m-1)\,(m-2)$
Number of intersections of nodal and cuspidal curves, stationary points on cuspidal curve ..... $50m^2-192m+168$
Number of intersections of nodal and cuspidal curves, stationary points on nodal curve ..... $12\,(m-3)\,(3m^3-6m^2-11m+18)$
Class of surface ..... m^2
Class of plane section ..... $3m\,(m-1)$
Number of its double tangents..... $\frac{1}{2}\,(m-1)\,(9m^3-9m^2-42m+48)$
„ „ inflexions ..... $3\,(m-1)\,(4m-5)$
Order of node couple curve ..... $3\,(m-1)^2\,(m^3+m-5)$
Class of node couple torse..... $\frac{m\,(m-1)}{2}\,(m^2+m-3)$
Number of its pinch planes ..... $6\,(m-1)^2$.

11. In a similar way we can obtain most of the characteristics of the surface represented by equating to zero the result of eliminating k homogeneously entering parameters $a_1, a_2 \ldots\ldots a_k$ from the system

$$\frac{dU}{da_1}=0,\ \frac{dU}{da_2}=0 \ \ldots\ldots\ \frac{dU}{da_k}=0,$$

where U is linear in x, y, z, w.

If the order of U in the parameters is m, we get

Order of surface ..... $k\,(m-1)^{k-1}$
Order of tangent cone ..... $\frac{k\,.\,k-1}{2}\,m\,(m-1)^{k-2}$
Class of cuspidal torse ... $\frac{k\,.\,k-1}{2\,.\,3}\,m\,(m-1)^{k-3}\,\{(k^2-1)\,m+k+3-2k^2\}$
Order of nodal curve ...

$$\tfrac{1}{2}k\,(m-1)^{k-2}\left(k(m-1)^k-\frac{3k^2+k-2}{2}\,(m-1)+\frac{3k^2-k-2}{2}\right)$$

Number of pinch points 0
Order of cuspidal curve ..... $\frac{k\,.\,k^2-1}{2}\,(m-1)^{k-2}\,(m-2)$
Class of surface ..... $\frac{k\,.\,k-1\,.\,k-2}{1\,.\,2\,.\,3}\,m^2\,(m-1)^{k-3}$.

From these can be determined $\delta', \kappa', \delta, \kappa, c', b$. To show how the above numbers may be established, I consider the class. The class is the order of the conditions for the coexistence of

$$\frac{dU}{da_1}=0 \ \ldots\ldots\ \frac{dU}{da_k}=0,\quad U'=0,\ U''=0,$$

where the accents denote that x, y, z, w are replaced by the same letters

accented like the functions. The order of the above system, by the general formula for a system having one common solution, is the number above given.

On some proposed forms of Slide-Rule. *By* G. H. Darwin.

[*Read March 11th*, 1875.]

The object of the author was to devise a form of slide-rule which should be small enough for the pocket, and yet be a powerful instrument.

The first proposed form was to have a pair of watch-spring tapes graduated logarithmically, and coiled on spring bobbins side by side. There was to be an arrangement for clipping the tapes together and unwinding them simultaneously. Two modifications of this idea were given.

The second form was explained as the logarithmic graduation of several coils of a helix engraved on a brass cylinder. On the brass cylinder was to fit a glass one, similarly graduated.

To avoid the parallax due to the elevation of the glass above the other scale, the author proposed that the glass cylinder might be replaced by a metal corkscrew sliding in a deep worm, by which means the two scales might be brought flush with one another.

April 8th, 1875.

Prof. H. J. S. SMITH, F.R.S., President, in the Chair.

Mr. J. H. Röhrs, M.A., was elected a Member, and Messrs. Nanson and Ritchie were admitted into the Society.

Mr. G. H. Darwin gave an account of two applications of Peaucellier's Cells: first, to "the Mechanical Description of Equipotential Lines"; and, secondly, to "a Mechanical Method of making a Force which varies inversely as the Square of the Distance from a Fixed Point." In this latter case, let O be the fixed pivot of a cell, and suppose the cell to be in equilibrium under the action of two forces P and P′ acting at B and D.

Then, by the principle of virtual velocities,

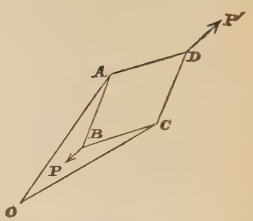
$$P'.\delta.OD + P.\delta.OB = 0.$$

Now, $OD.OB = OA^2 - AD^2$;

therefore $\frac{\delta.OD}{OD} = -\frac{\delta.OB}{OB}$;

therefore $P'.OD = P.OB$,

whence $P = P'\frac{OA^2 - AD^2}{OB^3}$.



If, then, P′ is a constant force acting away from O, P is an attractive force varying as OB^{-2}.

Mr. Darwin stated that the idea was the joint production of his brother, Horace Darwin, and himself, and that he entertained the hope that it would be possible to construct a toy to give an ocular proof of elliptic motion. Sir W. Thomson, F.R.S., having expressed himself much pleased with this latter application, as he had himself failed in some attempts he had made to do the same, then made two communications to the Society. The first was on "The Integration of the Equations for the motions of a system acted on by forces expressed by Linear Functions of the Displacements and Velocities"; the other "On the Vibrations of a Stretched String of Gyrostats (Dynamical Theory of Faraday's Magnetic Rotation of the Plane of Polarisation)."

Prof. Cayley made a few remarks "On some Integrals connected with the Theory of Attractions."

Mr. Tucker read a portion of a paper by Prof. Wolstenholme. The problem discussed in the paper is thus enunciated:—A tube of fine uniform bore is bent into the form of a regular polygon of n sides, and filled with equal volumes of n different fluids which do not mix; it is then closed, and held in any position in a vertical plane; the sides of the polygon formed by joining the common surfaces of the different fluids will always have constant directions; but if two conditions be satisfied, every position will be one of equilibrium. The author applies his result to a few simple cases: thus, if $n=3$, and the densities be in Arithmetical Progression, the straight line joining the ends of the fluid of mean density will always be vertical. Again, if $n=4$, and $\rho_1+\rho_4 = \rho_2+\rho_3$ (ρ_1, ρ_2, ρ_3, ρ_4 being the densities of the fluids), then the diagonals of the square formed by the surfaces of the fluids will be vertical and horizontal. This instrument, Prof. Wolstenholme suggests, might possibly be used as a level and plumb-line; perhaps also some interesting toys might be made by other polygons.

A paper by Prof. J. Clerk-Maxwell, F.R.S., "On the Application of Hamilton's Characteristic Function to Optical Instruments symmetrical about an Axis, and the Value of the Function for a Spherical Surface," was taken as read.

The following presents were received :—

"Annali di Matematica," Serie ii^{a}, Tom. vi., Fasc. 4^{o}.

"Bulletin des Sciences Mathématiques et Physiques," Dec. 1874. Tome vii. Also "Table des Matières" (1er Semestre 1874), Tome vi.

"Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich," redigirt von Dr. Rudolf Wolf. 17er Jahrgang, 1es, 2es, 3es, und 4es heft, 1872 ; 18er Jahrgang, 1es—4es heft, 1873.

"Cartes de Visite," from Messrs. W. H. Besant, F.R.S., J. Griffiths, and C. Smith.

The Mechanical Description of Equipotential Lines.

By G. H. Darwin.

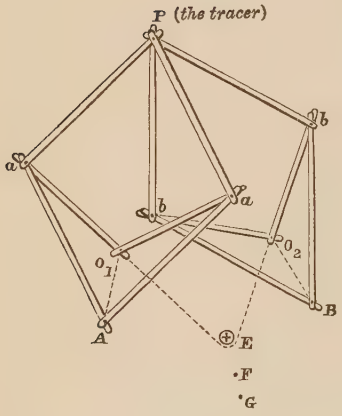
[*Read April* 8, 1875.]

Fig. 1 represents two Peaucellier's cells, of which O_1, O_2 are the respective fulcra, which are fixed ; A, P and B, P the poles. Let the moduli of the cells be respectively proportional to two charges of positive electricity at O_1 and O_2. Thus $O_1A = \frac{m_1}{O_1P}$, $OB = \frac{m_2}{O_2P}$, so that, if $O_1A + O_2B$ is constant, P traces the equipotential line of m_1 and m_2 placed at O_1 and O_2 respectively. The constancy of $O_1A + O_2B$ may be attained thus : Let the pivots at O_1, O_2, about which the cells turn, be two needles ; fasten a piece of pack-thread to A, pass it round O_1, round another needle E (driven into the drawing-board), round O_2, and fasten the other end to B. The broken line in fig. 1 represents this thread. Then, if P is moved so as to keep the thread taut, it describes one of the equipotential lines. If the needle E is shifted to F and G, we get another equipotential line.

Fig. 2 represents the arrangement of the thread where one of the charges is negative. In this and the succeeding figures, fixed points are marked with crosses, and the needles are exaggerated so as to show the disposition of the strings, and the bars of the cells are omitted, leaving only the tracing point P and the other poles marked. In this second case the requisite is that $OA \sim OB$ should be constant. This is obviously attained by making a knot K, or a loop in the string which passes round the fixed needle E. This knot or loop is held in one hand and pulled, whilst P, the tracing point, is released with the other, so as always to keep the strings taut. By shifting the knot or loop from point to point along the string, we get successive equipotentials.

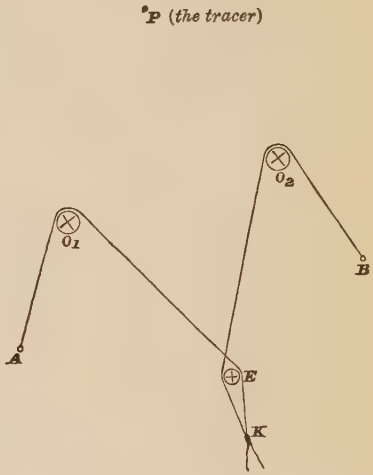
Some figures submitted to the Society were drawn with rough cells, the tracing point being the blunt end of needle ; and they appeared

FIG. 1.



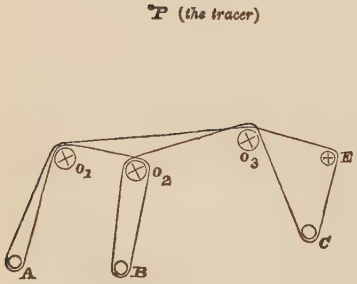
(Two positive charges.)

FIG. 2.



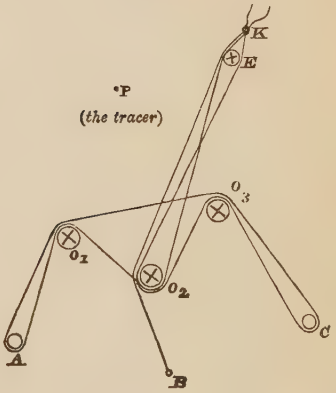
(One positive charge, and one negative.)

FIG. 3.



(Three positive charges.)

FIG. 4.



(Two positive charges, one negative.)

to the author sufficiently good to show that a carefully made instrument, would perform very well. Pack-thread was found to be but very slightly extensible.

In an attempt to draw with one of Mr. Hart's 4-bar reciprocators, the bars interfered much more with the fixed pivots than in the case of Peaucellier's cells; and with positive Peaucellier's cells but a small part of the field could be drawn.

Professor Sylvester has suggested to the author that it should be possible to do without string; and it would doubtless be possible, but it is to be doubted whether, as a practical point, any linkage would allow of the easy alteration of the constant which is permitted by the shifting of needle E, so as to draw successive lines quickly.

Figs. 3 and 4 show the arrangement of the string requisite for 3 points, either all positive, or two positive and one negative. No endeavour has been made to construct such an instrument, as it would require very careful workmanship; the string would have to be endless, and therefore smooth pegs would have to be fixed to the non-tracing poles of the cells. The field throughout which tracing would be possible would also be much restricted.

A similar kind of arrangement of strings is clearly theoretically applicable to any number of points, whatever were the signs of their charges.

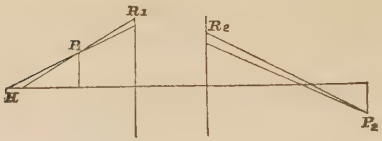
On the Application of Hamilton's Characteristic Function to the Theory of an Optical Instrument symmetrical about its axis. By J. Clerk-Maxwell.

[*Read April 8th*, 1875.]

When a ray of light passes from the point (x_1, y_1, z_1) to the point (x_2, y_2, z_2) through any series of media, the line-integral $V = \int \mu ds$ may be defined as the distance which light would travel in vacuum in the same time as it travels from (x_1, y_1, z_1) to (x_2, y_2, z_2).

Calling this the reduced distance between (x_1, y_1, z_1) and (x_2, y_2, z_2), Hamilton's Characteristic Function may be defined as the value of the reduced distance between two points expressed in terms of the coordinates of these points.

It is not necessary that the coordinates of the two points should be referred to the same system of axes. In treating of optical instruments we shall reckon z_1 and z_2 in opposite directions along the axis, and from different planes of reference. We shall, however, make the axes of x_1 and x_2 parallel to each other in what follows.



Let a ray from the point P_1 $(x_1, 0, z_1)$ pass through the first plane of reference at the point R_1 $(a_1, b_1, 0)$, through the second plane of reference at R_2 $(a_2, b_2, 0)$, and reach the point P_2 $(x_2, 0, z_2)$;

then, putting $\overline{P_1R_1}^2 = (x_1-a_1)^2 + b_1^2 + z_1^2 = r_1^2$

and $\overline{P_2R_2}^2 = (x_2-a_2)^2 + b_2^2 + z_2^2 = r_2^2$ } ..... (1),

and the characteristic function from R_1 to $R_2 = V$ (a function of a_1, b_1, a_2, b_2), the reduced path from P_1 to P_2 is

$$U = \mu_1 r_1 + V + \mu_2 r_2 \text{ .....} \quad (2).$$

This quantity is stationary with respect to variations of a_1, b_1, a_2, b_2; therefore, differentiating with respect to these variables, we get

$$\left.\begin{aligned} \mu_1 \frac{a_1-x_1}{r_1} + \frac{dV}{da_1} = 0, \quad \mu_1 \frac{b_1}{r_1} + \frac{dV}{db_1} = 0 \\ \mu_2 \frac{a_2-x_2}{r_2} + \frac{dV}{da_2} = 0, \quad \mu_2 \frac{b_2}{r_2} + \frac{dV}{db_2} = 0 \end{aligned}\right\} \text{ .....} \quad (3).$$

If P_1 and P_2 are conjugate foci for rays in the plane of xz, the reduced path is stationary in passing from one ray to the next by simultaneous variation of a_1 and a_2.

Hence

$$\left.\begin{aligned} \left[\mu_1\left(\frac{1}{r_1} - \frac{(a_1-x_1)^2}{r_1^3}\right) + \frac{d^2V}{da_1^2}\right] da_1 + \frac{d^2V}{da_1\,da_2}\,da_2 = 0 \\ \left[\mu_2\left(\frac{1}{r_2} - \frac{(a_2-x_2)^2}{r_2^3}\right) + \frac{d^2V}{da_2^2}\right] da_2 + \frac{d^2V}{da_1\,da_2}\,da_1 = 0 \end{aligned}\right\} \text{ .....} \quad (4).$$

Eliminating da_1 and da_2 from these equations, and putting θ for the angle between the ray and the axis, we find

$$\left(\frac{\mu_1}{r_1}\cos^2\theta_1 + \frac{d^2V}{da_1^2}\right)\left(\frac{\mu_2}{r_2}\cos^2\theta_2 + \frac{d^2V}{da_2^2}\right) = \left(\frac{d^2V}{da_1\,da_2}\right)^2 \text{ .....} \quad (5),$$

an equation connecting the values of r_1 and r_2 for conjugate primary focal lines formed by rays in the plane of xz.

For rays in a plane perpendicular to that of xz, we obtain the relation of r_1' and r_2' for the secondary focal lines, by passing from one ray to the next by simultaneous variation of b_1 and b_2,

$$\left.\begin{aligned}\left(\frac{\mu_1}{r_1}+\frac{d^2V}{db_1^2}\right)db_1+\frac{d^2V}{db_1\,db_2}db_2=0\\ \left(\frac{\mu_2}{r_2'}+\frac{d^2V}{db_2^2}\right)db_2+\frac{d^2V}{db_1\,db_2}db_1=0\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(6),$$

whence

$$\left(\frac{\mu_1}{r_1'}+\frac{d^2V}{db_1^2}\right)\left(\frac{\mu_2}{r_2'}+\frac{d^2V}{db_2^2}\right)=\left(\frac{d^2V}{db_1\,db_2}\right)^2\quad\ldots\ldots\ldots\ldots\ldots\ldots(7).$$

The form of this equation differs from that for the primary focal lines only by the omission of $\cos^2\theta$, and by substituting b for a.

From these equations we obtain the following values of the cardinal points of the ray in the primary and seondary plane:

$$\left.\begin{aligned}&\text{Let}\quad A=\frac{d^2V}{da_1^2},\quad B=\frac{d^2V}{da_1\,da_2},\quad C=\frac{d^2V}{da_2^2},\quad D=\frac{1}{B^2-AC}\\ &\qquad A'=\frac{d^2V}{db_1^2}\quad B'=\frac{d^2V}{db_1\,db_2},\quad C'=\frac{d^2V}{db_2^2},\quad D'=\frac{1}{B'^2-A'C'}\end{aligned}\right\}\ldots(8).$$

	Primary.	Secondary.
Value of r_1 at 1st principal focus	$g_1=\mu_1 CD\cos^2\theta_1$	$g_1'=\mu_1 C'D'$
Value of r_2 at 2nd principal focus	$g_2=\mu_2 AD\cos^2\theta_2$	$g_2'=\mu_2 A'D'$
1st principal focal length	$f_1=\mu_1 BD\cos\theta_1\cos\theta_2$	$f_1'=\mu_1 B'D'$
2nd principal focal length	$f_2=\mu_2 BD\cos\theta_1\cos\theta_2$	$f_2'=\mu_2 B'D'$

(9).

In tracing the axis of the pencil, we may, as a first approximation, neglect the square of θ.

Let it cross the axis before incidence at H_1, and after emergence at H_2, where H_1 is $\frac{1}{\alpha}f_1$ beyond the first principal focus, and H_2 is αf_2 beyond the second principal focus.

Let it cut the first and second principal planes at a distance ξ from the axis. Then, since these planes are at distances f_1 and f_2 within the principal foci,

$$\tan\theta_1=\frac{\xi}{f_1}\,\frac{\alpha}{1+\alpha},\qquad \tan\theta_2=\frac{\xi}{f_2}\,\frac{1}{1+\alpha}\quad\ldots\ldots\ldots\ldots(10),$$

$$\alpha_1=\frac{\xi}{1+\alpha}\left(\frac{C}{B}\alpha+1\right),\qquad \alpha_2=\frac{\xi}{1+\alpha}\left(\frac{A}{B}+\alpha\right)\ldots\ldots\ldots(11).$$

Let the object and image be respectively $\frac{1}{\beta}f_1$ and βf_1 beyond the principal foci,

$$z_1=f_1\left(\frac{C}{B}+\frac{1}{\beta}\right),\quad z_2=f_2\left(\frac{A}{B}+\beta\right)\quad\ldots\ldots\ldots\ldots(12),$$

$$x_1=\frac{\xi}{1+\alpha}\left(1-\frac{\alpha}{\beta}\right),\ x_2=\frac{\xi}{1+\alpha}(\alpha-\beta)\quad\ldots\ldots\ldots\ldots(13).$$

We shall next determine the relation between the curvature of the object and that of the image.

Let their curvatures be concave towards the instrument, their radii being R_1 and R_2 respectively, then,

$$z_1 = f_1\left(\frac{C}{B}+\frac{1}{\beta}\right) - \frac{x_1^2}{2R_1} + \&c., \quad z_2 = f_2\left(\frac{A}{B}+\beta\right) - \frac{x_2^2}{2R_1} + \&c. \ldots (14).$$

We must now insert these values in equation (5), but, for the sake of simplicity, we shall suppose the planes of reference to pass through the principal foci.

We now find

$$f_1 = \frac{\mu_1}{B},\ f_2 = \frac{\mu_2}{B} \quad \ldots\ldots\ldots\ldots (15),$$

$$z_1 = \frac{1}{\beta} f_1 - \frac{x_1^2}{2R_1}, \quad z_2 = \beta f_2 - \frac{x_2^2}{2R_1} \quad \ldots\ldots\ldots\ldots (16),$$

$$a_1 = \frac{\xi}{1+a}, \qquad a_2 = \frac{\xi a}{1+a} \quad \ldots\ldots\ldots\ldots (17).$$

Hamilton's function for a symmetrical instrument is of the form

$$\begin{aligned} V = {} & \tfrac{1}{2}\mathfrak{A}\,(a_1^2+b_1^2) + \mathfrak{B}\,(a_1a_2+b_1b_2) + \tfrac{1}{2}\mathfrak{C}\,(a_2^2+b_2^2) \\ & + \tfrac{1}{4}a\,(a_1^2+b_1^2)^2 + \tfrac{1}{2}b\,(a_1a_2+b_1b_2)^2 + \tfrac{1}{4}c\,(a_2^2+b_2^2)^2 \\ & + p\,(a_1a_2+b_1b_2)\,(a_2^2+b_2^2) + \tfrac{1}{2}q\,(a_1^2+b_1^2)\,(a_2^2+b_2^2) \\ & + r\,(a_1^2+b_1^2)\,(a_1a_2+b_1b_2) \quad \ldots\ldots\ldots\ldots (18). \end{aligned}$$

Differentiating, and making b_1 and b_2 zero, we find

$$\left.\begin{aligned} \mathfrak{A} &= \mathfrak{A} + 3aa_1^2 + 6ra_1a_2 + (b+q)\,a_2^2 \\ \mathfrak{B} &= \mathfrak{B} + 3ra_1^2 + (2b+2q)\,a_1a_2 + 3pa_2^2 \\ \mathfrak{C} &= \mathfrak{C} + (b+q)\,a_1^2 + 6pa_1a_2 + 3ca_2^2 \end{aligned}\right\} \quad \ldots\ldots\ldots (19),$$

$$\left.\begin{aligned} \mathfrak{A}' &= \mathfrak{A} + aa_1^2 + 2ra_1a_2 + qa_2^2 \\ \mathfrak{B}' &= \mathfrak{B} + ra_1^2 + ba_1a_2 + pa_2^2 \\ \mathfrak{C}' &= \mathfrak{C} + qa_1^2 + 2paa_2^2 + ca_2^2 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (20).$$

If the planes of reference pass through the principal foci,

$$\mathfrak{A} = 0 \quad \text{and} \quad \mathfrak{C} = 0 \quad \ldots\ldots\ldots\ldots (21).$$

Substituting in equations (5) and (7), and putting

$$U = \frac{2}{(a-\beta)^2}\left\{\begin{aligned} & \mathfrak{B}'\beta\left(\frac{a^2}{\mu_1^2}+\frac{1}{\mu_2^2}\right) - \frac{1}{\mathfrak{B}^2}\left[a+\tfrac{1}{2}b\,(a-\beta)^2 + ca^2\beta^2\right. \\ & \left. -2pa\beta\,(a-\beta) - 2qa\beta + 2r(a-\beta)\right] \end{aligned}\right\} \quad \ldots\ldots (22),$$

$$W = 2\left(\frac{a+\beta}{a-\beta}\right)^2 \frac{1}{\mathfrak{B}^2}\left(\tfrac{1}{2}b-q\right) \quad \ldots\ldots\ldots\ldots (23),$$

we find

$$\frac{1}{\mu_1 R_1} + \frac{1}{\mu_2 R_2} = 3U + W \quad \text{for the primary image} \ldots (24),$$

$$\frac{1}{\mu_1 R_1'} + \frac{1}{\mu_2 R_2'} = U + W \quad \text{for the secondary image} \ldots (25).$$

The condition of distinctness is $U=0$.

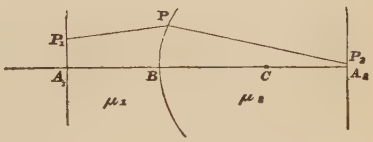
That the image of a flat object may be flat as well as distinct,

$$U = 0 \quad \text{and} \quad W = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (26).$$

Form of the Characteristic Function for a Spherical Refracting Surface.

Let the planes of reference pass through the principal foci.

Let $BC = s$ be the radius, then $AB = \frac{\mu_1 s}{\mu_2 - \mu_1}$, $BA_2 = \frac{\mu_2 s}{\mu_2 - \mu_1}$.



If the coordinates of P_1 are $a_1, b_1, -\frac{\mu_1 s}{\mu_2 - \mu_1}$,

P_2 „ $a_2, b_2, \frac{\mu_2 s}{\mu_2 - \mu_1}$

P „ a, b, z,

and if $P_1P = r_1$ and $PP_2 = r_2$, then

$$V = \mu_1 r_1 + \mu_2 r_2,$$

under the condition that V is stationary with respect to variations of a and b, when $a^2 + b^2 + (s-z)^2 = s^2$.

This gives the following values of a and b:

$$a = a_1 + a_2 + \lambda_1 a_1 + \lambda_2 a_2 + \text{ \&c.},$$

$$b = b_1 + b_2 + \lambda_1 b_1 + \lambda_2 b_2 + \text{ \&c.},$$

when $\lambda_1 = \frac{1}{2s^2\mu_1^2\mu_2^2}\{\mu_1^3(a_1^2+b_1^2) + 2\mu_1^2\mu_2(a_1a_2+b_1b_2) + \mu_2(\mu_1^2-\mu_1\mu_2+\mu_2^2)(a_2^2+b_2^2)\}$,

$\lambda_2 = \frac{1}{2s^2\mu_1^2\mu_2^2}\{\mu_1(\mu_1^2-\mu_1\mu_2+\mu_2^2)(a_1^2+b_1^2) + 2\mu_1\mu_2^2(a_1a_2+b_1b_2) + \mu_2^3(a_2^2+b_2^2)\}$;

whence we get, as the value of V to terms of the fourth order,

$$V = s\frac{\mu_1^2+\mu_2^2}{\mu_2-\mu_1} - \frac{1}{s}(\mu_2-\mu_1)(a_1a_2+b_1b_2)$$

$$+ \frac{\mu_2-\mu_1}{8s^3\mu_1^2\mu_2^2}\left\{\begin{array}{l}\mu_1^2(3\mu_1^2+\mu_2^2)(a_1^2+b_1^2)^2 + 8\mu_1\mu^2(\mu_1^2+\mu_2^2)(a_1a_2+b_1b_2)^2 \\ +\mu_2^2(\mu_1^2+3\mu_2^2)(a_2^2+b_2^2)^2 \\ +4\mu_2(\mu_1^3+2\mu_1\mu_2^2+\mu_3^2)(a_1a_2+b_1b_2)(a_2^2+b_2^2) \\ +\mu_1\mu_2(7\mu_1^2-6\mu_1\mu_2+7\mu_2^2)(a_1^2+b_1^2)(a_2^2+b_2^2) \\ +4\mu_1(\mu_1^3+2\mu_1^2\mu_2+\mu_2^3)(a_1^2+b_1^2)(a_1a_2+b_1b_2)\end{array}\right\}$$

Hence the coefficients in the general equations (18), (19), (20), (22), (23) are as follows:—

$$\mathfrak{A} = 0, \qquad \mathfrak{B} = \frac{\mu_1 - \mu_1}{s}, \qquad \mathfrak{C} = 0,$$

$$a = \tfrac{1}{2}\frac{\mu_2 - \mu_1}{s^3\mu_2^2}(3\mu_1^2 + \mu_2^2), \quad b = 2\frac{\mu_2 - \mu_1}{s^3\mu_1\mu_2}(\mu_1^2 + \mu_2^2), \quad c = \tfrac{1}{2}\frac{\mu_2 - \mu_1}{s^3\mu_1^2}(\mu_1^2 + 3\mu_2^2),$$

$$p = \tfrac{1}{2}\frac{\mu_2 - \mu_1}{s^3\mu_1^2\mu_2}(\mu_1^3 + 2\mu_1\mu_2^2 + \mu_2^3), \quad q = \tfrac{1}{4}\frac{\mu_2 - \mu_1}{s^3\mu_1\mu^2}(7\mu_1^2 - 6\mu_1\mu_2 + 7\mu_2^2),$$

$$r = \tfrac{1}{2}\frac{\mu_2 - \mu_1}{s^3\mu_1\mu_2^2}(\mu_1^3 + 2\mu_1^2\mu_2 + \mu_2^3).$$

May 13th, 1875.

Prof. CAYLEY, F.R.S., Vice-President, in the Chair.

Messrs. A. B. Kempe and C. Taylor were admitted into the Society. Messrs. E. A. Hunter, E. H. Rhodes, and W. J. Wright (U.S.) were proposed for election.

The following papers were read:—

Rev. C. Taylor, "On some Constructions for transforming Curves and Surfaces;" Mr. J. W. L. Glaisher, "Notes on Laplace's Coefficients." Dr. Hirst and the Chairman made remarks on the former of these papers, and the Chairman again upon the latter paper. A communication from Mr. Harry Hart, "On a Linkwork for describing Spheroconics and Sphero-quartics," was taken as read.

The following presents were received:—

"Bulletin des Sciences Mathématiques et Physiques," Janvier, 1875.

"Monatsbericht," Januar, Februar, 1875.

"Sui Determinanti di Funzioni," nota del Prof. Felice Casorati.

"Alcune formole fondamentali per lo studio delle equazioni algebrico differenziali di primo ordine e secondo grado tra due variafili ad integrale generale algebrica," nota del Prof. F. Casorati.

"Société des Sciences physiques et naturelles de Bordeaux aux Facultés (Rue Mombazon, N° 4) — Extrait des procès verbaux des Séances.

"Register für die Monatsberichte vom Jahre 1859 bis 1873." Berlin, 1875.

"Crelle", achtzigster Band, zweites Heft.

"Reprint of Mathematics from *Educational Times*," vols. xxi. xxii.: from Messrs. Hodgson & Son.

Journal de l'École Polytechnique, 44$^{\text{ieme}}$ cahier, tome xxvii., 1874: from the General Commandant l'École Polytechnique.

"Proceedings of the Royal Society," vols. 154, 155, 158—161.

"On the Geometrical Representation of some familiar Cases of Reaction in Rigid Dynamics," by Prof. R. Townsend, F.R.S. ("Quarterly Journal of Mathematics," No. 51): from the author.

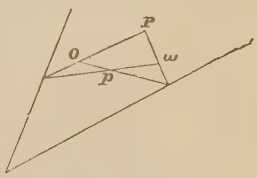
The Method of Reversion applied to the Transformation of Angles.

By C. Taylor.

[*Abstract of Paper read May 13th*, 1875.]

The basis of this paper is a neglected work on Conic Sections (by G. Walker, F.R.S., Nottingham, 1794), which, for originality and thoroughness, is, in its own special department, unsurpassed.

Walker establishes a connexion between a conic and a circle by means of a homographic transformation, which is a particular case of the following:—

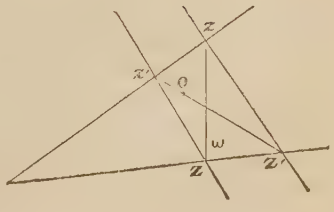


Take fixed origins O, ω, and corresponding to each take a fixed straight line or axis, and let the law of correspondence between points, as P, p, be that PO, $p\omega$ intersect on the ω-axis, and Pω, pO on the O-axis. Call P, p REVERSE points. Then it is evident that each origin is the reverse of the other; that reverse curves are of the same degree; and, generally, that the relations of any point to either origin are of the same character as those of the reverse point to the reverse origin.

To determine the limits of deformation, take origins O, ω, and let it be required to determine the corresponding axes. Assume that a given point p shall be the reverse of a given point P. Then since PO, ωp meet on the ω-axis, and Pω, Op on the O-axis, one point on each axis is known. In like manner, by reversing a point Q into q, a second point on each axis is determined. Hence, in general, it is possible to reverse a quadrilateral OPQω into a quadrilateral ωpqO.

Construction for Reverse Lines.

It is evident that the reverse of a point z on the ω-axis is the point Z in which $z\omega$ meets the O-axis; and the reverse of a point Z′ on the O-axis is the point z' in which Z′O meets the ω-axis. Hence the straight line Zz' is the reverse of zZ′.



Taking the axes of reversion as axes of coordinates, and (h, k), (H, K) as the coordinates of the origins of reversion, it is easy to see that reverse lines, as zZ', Zz', may be represented by equations of the forms

$$\frac{x}{a} + \frac{y}{b} - 1 = 0 \quad \text{.....(A),}$$

$$\frac{x}{\mathrm{H}} + \frac{y}{k} - 1 = \frac{\mathrm{K}}{\mathrm{H}} \cdot \frac{x}{b} + \frac{h}{k} \cdot \frac{y}{a} \quad \text{..... (B).}$$

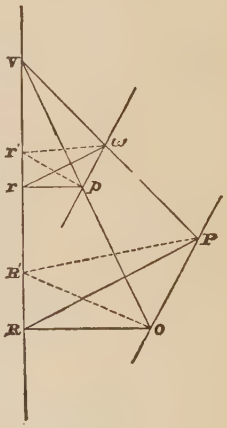
Let a, b vary proportionally. Then (A) represents a system of parallels, and (B) a system of straight lines intersecting on

$$\frac{x}{\mathrm{H}} + \frac{y}{k} = 1 \quad \text{.....(C),}$$

which is therefore reverse to the line at infinity. The line (C) might have been determined by removing Z′ to infinity on the O-axis, and z to infinity on the ω-axis.

To proceed to the special case of reversion with which I am chiefly concerned. Let the ω-axis be at infinity, and let the O-axis be called the Base Line. We have then the construction for reverse points P, p :—

Through the origins of reversion O, ω *draw straight lines meeting, viz. in* V, *on the base line; and let* Vω, VO *be intersected in* P, p *by a pair of parallels through* O, ω.



Reverse Lines PR, pr meet the base line in points R, r, such that PR, OR are parallel respectively to ωr, pr.

The base line is the reverse of the line at infinity.

Hence the law of the

Reversion of Angles.

If the straight lines containing an angle P *meet the base line in* R,R′, *the lines containing the reverse angle* p *intercept on the base line a length* rr' *which subtends at* ω *an angle equal to* P.

Examples of the Reversion of Angles.

1. The Ortho-centre.—Let the sides of a triangle abc, the reverse of ABC, meet the base line in α, β, γ; draw through ω lines at right angles to $\omega\alpha$, $\omega\beta$, $\omega\gamma$ to meet the base line in a', b', c'. Then it is easily seen that aa', bb', cc' are reverse to the perpendiculars of the triangle ABC, and therefore *co-intersect* in a point θ, the reverse of the ortho-centre of ABC.

2. If the triangle *abc* envelopes a fixed conic which *touches the base line*, the point θ moves on a straight line, since the reverse triangle ABC envelopes a parabola, and its orthocentre therefore moves on a fixed straight line, viz. the directrix.

It is easy to deduce, that if a triangle *abc* envelopes a circle, and if the three parallel tangents meet a seventh tangent in a', b', c'; then aa', bb', cc' are parallel.

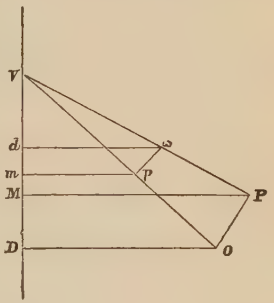
3. *Angles subtended at* O, ω *are equal each to each*, in consequence of the parallelism of OP, $p\omega$. To illustrate this special case, take the theorem that

A chord of a conic which subtends a right angle at a fixed point on the curve, passes through a fixed point on the normal;

which follows by reversion from the fact that the angle in a semicircle is a right angle, if this be first expressed in the form:

A chord of a circle which subtends a right angle at a fixed point on the circumference, passes through a fixed point on the normal.

WALKER'S CIRCLE.



Let fall perpendiculars ωd, OD, PM, pm upon the base line. Then, by parallels,

$$\mathrm{OP} : \omega p = \mathrm{PV} : \omega \mathrm{V} = \mathrm{PM} : \omega d.$$

Hence, if the locus of p be a circle about ω, the locus of P will be a conic having O for focus and the base line for directrix.

It is on this property that Walker's system of Geometrical Conics is based. See "Messenger of Mathematics," Vol. ii. p. 97 (1872).

It may be shewn that

$$\mathrm{PM} \,.\, pm = \mathrm{OD} \,.\, \omega d,$$

and hence that, if $p\ (x, y)$ be referred to ωd and the base line as axes, and P (X, Y) be referred to OD and the base line as axes, the analytical reverse transformation will be

$$y = \frac{\omega d \,.\, \mathrm{OD}}{\mathrm{Y}}; \quad -x = \frac{\omega d \,.\, \mathrm{X}}{\mathrm{Y}}.$$

Compare Newton's "Principia," Lib. I., Lemma 22: "Figuras in alias ejusdem generis figuras mutare."

The method of *Reversion* is easily adapted to transformation in space.

Notes on Laplace's Coefficients. *By* J. W. L. Glaisher, M.A.

[*Read May* 13*th*, 1875.]

1. The present paper merely consists of a few detached notes having relation to Laplace's (or, more correctly, Legendre's) coefficients; the n^{th} Laplace's coefficient, P_n, being defined as the coefficient of h^n in the expansion of

$$\frac{1}{\sqrt{(1-2ha+h^2)}}$$

in ascending powers of h.

2. It is natural to examine to what results we are led by expressing $(1-2ha+h^2)^{-\frac{1}{2}}$ as a definite integral, and then expanding in powers of h. For this purpose there are three formulæ available, viz.:—

$$\frac{\pi}{\sqrt{(a^2-b^2)}} = \int_0^\pi \frac{dt}{a+b\cos t} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$\frac{\sqrt{\pi}}{2\sqrt{a}} = \int_0^\infty e^{-at^2}dt \quad \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$\frac{\pi}{2\sqrt{a}} = \int_0^\infty \frac{dt}{a+t^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

The first of these has been discussed by Jacobi (Crelle, t. xxvi., p. 81); it leads at once* to the formula given by Laplace ("Mécanique Céleste," 1825, t. v. p. 33), viz.,—

$$\pi \,.\, P_n = \int_0^\pi \{a-\cos t\,\sqrt{(a^2-1)}\}^n dt \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

From the second we have

$$\sqrt{\pi} \,.\, P_n = \text{coefficient of } h^n \text{ in } 2\int_0^\infty e^{-(1-a^2)t^2-(h-a)^2t^2}dt$$

$$= \frac{2}{n\,!}\int_0^\infty e^{-(1-a^2)t^2}\left(\frac{d}{dh}\right)^n e^{-(h-a)^2t^2}dt_{(h=0)}$$

$$= \frac{2}{n\,!}\int_0^\infty e^{-(1-a^2)t^2}\left(-\frac{d}{da}\right)^n e^{-a^2t^2}dt,$$

which we may write

$$\sqrt{\pi} \,.\, P_n = \frac{1}{n\,!}\int_{-\infty}^\infty e^{-(1-a^2)t^2}\left(-\frac{d}{da}\right)^n e^{-a^2t^2}\,dt \quad \ldots\ldots\ldots (5),$$

and is the form to which (2) leads.

* See Meyer, "Vorlesungen über die Theorie der bestimmten Integrale" (Leipzig, 1871), p. 407.

Formula (3) gives

$$\frac{\pi}{2\sqrt{(1-2ha+h^2)}} = \int_0^\infty \frac{dt}{(h-a)^2+\gamma^2} \text{ if } \gamma = \sqrt{(1-a^2+t^2)}$$

$$= \int_0^\infty \frac{dt}{2\gamma i}\left(\frac{1}{h-a-\gamma i} - \frac{1}{h-a+\gamma i}\right),$$

whence

$$\pi \,.\, P_n = \frac{1}{i}\int_0^\infty \frac{dt}{\sqrt{(1-a^2+t^2)}}\left\{\frac{1}{[a-i\sqrt{(1-a^2+t^2)}]^{n+1}} - \frac{1}{[a+i\sqrt{(1-a^2+t^2)}]^{n+1}}\right\} \quad ...(6).$$

3. The integral (2) can be made to give Ivory's expression for P_n without difficulty. We have

$$\sqrt{\pi} \,.\, P_n = \text{coefficient of } h^n \text{ in } \int_{-\infty}^{\infty} e^{-(1-a^2)t^2-(h-a)^2t^2}dt$$

$$= \quad ,, \quad ,, \quad \int_{-\infty}^{\infty} e^{-(1-a^2)t^2-(h-at)^2} t^n dt$$

$$= \quad ,, \quad ,, \quad e^{-h^2}\int_{-\infty}^{\infty} e^{-t^2+2hat}\, t^n dt$$

$$= \quad ,, \quad ,, \quad \frac{e^{-h^2}}{2^n h^n}\left(\frac{d}{da}\right)^n \int_{-\infty}^{\infty} e^{-t^2+2hat}\, dt$$

$$= \frac{1}{2^n} \cdot \text{coefficient of } h^{2n} \text{ in } e^{-h^2}\left(\frac{d}{da}\right)^n \cdot \sqrt{\pi}\, e^{a^2h^2};$$

$$\text{viz., in } \sqrt{\pi}\cdot\left(\frac{d}{da}\right)^n e^{(a^2-1)h^2},$$

whence

$$P_n = \frac{1}{2^n \,.\, n\,!}\left(\frac{d}{da}\right)^n (a^2-1)^n,$$

which is the value of P_n given by Ivory ("Phil. Trans.," 1824, p. 93).

The above investigation is chiefly interesting because, as a rule, it is difficult to obtain by a direct process a result of the form $\left(\frac{d}{dx}\right)^n X^n$ without the aid of Lagrange's theorem or the previous expansion of X^n.

4. In order to identify (5) with the ordinary expanded value of P_n, viz.,

$$P_n = \frac{1\,.\,3\,.\,5\ldots 2n-1}{n!}\left(a^n - \frac{n\,.\,n-1}{2\,.\,2n-1}\, a^{n-2} + \frac{n\,.\,n-1\,.\,n-2\,.\,n-3}{2\,.\,4\,.\,2n-1\,.\,2n-3}\, a^{n-4} - \&c.\right) \ldots\ldots\ldots(7),$$

it is necessary to find the expansion of $e^{a^2t^2}\left(\frac{d}{da}\right)^n e^{-a^2t^2}$. This may be obtained in two ways, viz., by means of a definite integral, or by finding the coefficient of h^n in $e^{(a+h)^2}$.

Starting with $\quad \sqrt{\pi}\,.\,e^{a^2} = \int_{-\infty}^{\infty} e^{-x^2+2ax}\,dx,$

we have $\sqrt{\pi}\,.\left(\frac{d}{da}\right)^n e^{a^2} = 2^n \int_{-\infty}^{\infty} e^{-x^2+2ax} x^n dx$

$$= 2^n e^{a^2} \int_{-\infty}^{\infty} e^{-(x-a)^2} x^n\,dx = 2^n e^{a^2} \int_{-\infty}^{\infty} e^{-x^2} (x+a)^n\,dx,$$

whence

$$e^{-a^2}\left(\frac{d}{da}\right)^n e^{a^2} = \frac{2^n}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} \left\{ a^n + n\,.\,a^{n-1}x + \frac{n\,.\,n-1}{1\,.\,2}\,a^{n-2}x^2 + \&c. \right\}$$

$$= 2^n \left\{ a^n + \frac{1}{2}\,.\,\frac{n\,.\,n-1}{1\,.\,2}\,a^{n-2} + \frac{1}{2}\,.\,\frac{3}{2}\,.\,\frac{n\,.\,n-1\,.\,n-2\,.\,n-3}{1\,.\,2\,.\,3\,.\,4}\,a^{n-4} + \&c. \right\} \quad \ldots\ldots\ldots(8).$$

Or we may proceed thus,

$$\left(\frac{d}{da}\right)^n e^{a^2} = n!\,.\,\text{coefficient of } h^n \text{ in } e^{(a+h)^2},$$

whence

$$e^{-a^2}\left(\frac{d}{da}\right)^n e^{a^2} = n!\,.\,\text{coefficient of } h^n \text{ in } 1 + h\,(2a+h) + \frac{h^2}{1\,.\,2}(2a+h)^2 + \&c.$$

$$= \text{coefficient of } h^n \text{ in}$$

$$h^n\,(2a+h)^n + nh^{n-1}(2a+h)^{n-1} + \frac{n\,.\,n-1}{1\,.\,2}\,h^{n-2}\,(2a+h)^{n-2} + \&c.$$

$$= (2a)^n + n\,.\,n-1\,(2a)^{n-2} + \frac{n\,.\,n-1\,.\,n-2\,.\,n-3}{1\,.\,2}\,(2a)^{n-4} + \&c. \ldots\ldots(9),$$

which is readily identified with (8).

Writing the binomial theorem coefficients n, $\frac{n\,.\,n-1}{1\,.\,2}$, ... as $B_1, B_2, \ldots$ and replacing a by iat in (8), we find

$$e^{a^2t^2}\left(-\frac{d}{da}\right)^n e^{-a^2t^2} = 2^n t^n \left\{ a^n t^n - \tfrac{1}{2} B_2 a^{n-2} t^{-2} + \tfrac{1}{2}\,.\,\tfrac{3}{2}\,B_4 a^{n-4} t^{n-4} - \&c. \right\},$$

whence

$$\int_{-\infty}^{\infty} e^{-t^2}\,.\,e^{a^2t^2}\left(-\frac{d}{da}\right)^n e^{-a^2t^2}\,dt = 2^n \left\{ a^n\,.\,\Gamma\left(\frac{2n+1}{2}\right) - \tfrac{1}{2} B\; a^{n-2}\,.\,\Gamma\left(\frac{2n-1}{2}\right) \right.$$

$$\left. + \tfrac{1}{2}\,.\,\tfrac{3}{2} B_4 a^{n-4}\,.\,\Gamma\left(\frac{2n-3}{2}\right) - \&c. \right\},$$

which agrees with (7), and affords a verification of (5).

5. In connexion with the integral

$$\int_{-\infty}^{\infty} e^{-ax^2 \pm 2bx}\, dx = \frac{\sqrt{\pi}}{\sqrt{a}}\, e^{\frac{b^2}{a}},$$

which has been used in the last section, it is worth noticing that it gives rise to a curious relation between the differentials of e^{a^2} and ae^{a^2}. For we have

$$\left(\frac{d}{2db}\right)^{2n} \frac{e^{\frac{b^2}{a}}}{\sqrt{a}} = \left(-\frac{d}{da}\right)^{n} \frac{e^{\frac{b^2}{a}}}{\sqrt{a}},$$

which becomes, on replacing a by a^{-2},

$$\left(\frac{d}{db}\right)^{2n} e^{a^2b^2} = \frac{1}{a}\left(2a^3\frac{d}{da}\right)^{n} ae^{a^2b^2}.$$

Now, generally, $\quad a^n\left(\frac{d}{da}\right)^{n}\phi(ab) = b^n\left(\frac{d}{db}\right)^{n}\phi(ab)\,;$

so that $\quad \frac{a^{2n}}{b^{2n}}\left(\frac{d}{da}\right)^{2n} e^{a^2b^2} = \frac{1}{a}\left(2a^3\frac{d}{da}\right)^{n} ae^{a^2b^2},$

or, putting $b = 1$,

$$a^{2n+1}\left(\frac{d}{da}\right)^{2n} e^{a^2} = \left(2a^3\frac{d}{da}\right)^{n} ae^{a^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (10),$$

the relation in question.

In the same manner, from the integral

$$\int_0^{\infty} e^{-ax^2-2bx}\, dx = \frac{1}{\sqrt{a}}\, e^{\frac{b^2}{a}}\, \mathrm{Erfc}\,\frac{b}{\sqrt{a}},$$

where $\quad \mathrm{Erfc}\, x = \int_x^{\infty} e^{-x^2}\, dx,$

we find $\quad \frac{a^{2n}}{b^{2n}}\left(\frac{d}{da}\right)^{2n} e^{a^2b^2}\,\mathrm{Erfc}\, ab = \frac{1}{a}\left(2a^3\frac{d}{da}\right)^{n} ae^{a^2b^2}\,\mathrm{Erfc}\, ab \ldots\ldots\ldots (11),$

or, putting $b = 1$,

$$a^{2n+1}\left(\frac{d}{da}\right)^{2n} e^{a^2}\,\mathrm{Erfc}\, a = \left(2a^3\frac{d}{da}\right)^{n} ae^{a^2}\,\mathrm{Erfc}\, a \ldots\ldots\ldots (12).$$

The formula (10) follows from (11) by changing the sign of b and adding, since

$$\mathrm{Erfc}\, x + \mathrm{Erfc}\,(-x) = \sqrt{\pi}.$$

6. Laplace's formula,

$$\pi \,.\, \mathrm{P}_n = \int_0^{\pi} \{a \pm \cos t\, \sqrt{(a^2-1)}\}^n\, dt,$$

leads at once to a simple relation between the coefficients of different

orders, which it is just possible may not have been remarked. For, E being a symbol of operation such that $EP_n = P_{n+1}$, then

$$\pi \cdot e^{-h\mathrm{E}} P_0 = \int_0^\pi e^{-ha - h\cos t \sqrt{(a^2-1)}}\, dt,$$

whence
$$\pi e^{-h(\mathrm{E}-a)} P_0 = \int_0^\pi e^{-h\cos t \sqrt{(a^2-1)}}\, dt;$$

or, equating the coefficients of h^n,

$$\pi (\mathrm{E}-a)^n P_0 = (a^2-1)^{\frac{1}{2}n} \int_0^\pi \cos^n t\, dt;$$

so that $P_n - na P_{n-1} + \frac{n \cdot n-1}{1 \cdot 2} a^2 P_{n-2} \ldots \pm a^n P_0 = 0$, if n be uneven

and $= \frac{1 \cdot 3 \cdot 5 \ldots n-1}{2 \cdot 4 \cdot 6 \ldots n}$ if n be even (13).

Examples: $n=2$, $\frac{1}{2}(3a^2-1) - 2a \cdot a + a^2 = \frac{1}{2}(a^2-1)$;

$n=3$, $\frac{1}{2}(5a^3-3a) - 3a \cdot \frac{1}{2}(3a^2-1) + 3a^2 \cdot a - a^3 = 0$.

7. The formula (13), found in the last section, would not of course be suitable for the actual calculation of P_n, as P_n is given in terms of P_{n-1}, $P_{n-2} \ldots P_0$. The usual formulæ which are most convenient for the calculation of P_n are

$$P_n = a P_{n-1} - \frac{1-a^2}{n} \frac{dP_{n-1}}{da} \quad \ldots\ldots\ldots\ldots\ldots\ldots (14),$$

and
$$P_n = \frac{2n-1}{n} a P_{n-1} - \frac{n-1}{n} P_{n-2} \ldots\ldots\ldots\ldots\ldots (15);$$

but a third formula, almost as convenient as either, was given by Mr. R. Kalley Miller, in his problem paper in the Mathematical Tripos, 1871 (Thursday morning), viz., that

$$P_n = a P_{n-1} + (n-1) \int_0^a P_{n-1}\, da + C \ldots\ldots\ldots\ldots\ldots (16),$$

where $C = 0$ or $(-)^{\frac{1}{2}n} \frac{n!}{2^n (\frac{1}{2}n!)^2}$, according as n is uneven or even.

Mr. Miller's formula is easily deduced from (14) by means of the equation

$$\frac{d}{da}\left\{(1-a^2)\frac{dP_{n-1}}{da}\right\} + n - 1 \cdot n P_{n-1} = 0,$$

which gives
$$-\frac{1-a^2}{n} \frac{dP_{n-1}}{da} = (n-1) \int_0^a P_{n-1}\, da + C,$$

C being so chosen that the two sides of this equation shall agree when $a=0$, that is to say, that C shall be zero if n be uneven, but, if n be

even, shall be equal to

$$-\frac{1}{n}\text{ coefficient of } h^{n-1} \text{ in } \frac{d}{da}\frac{1}{\sqrt{(1-2ha+h^2)}}\ (a=0);$$

$$=-\frac{1}{n}\text{ coefficient of } h^{n-2} \text{ in } (1+h^2)^{-\frac{3}{2}}$$

$$=\frac{1}{n}\cdot(-)^{\frac{1}{2}n}\frac{\frac{3}{2}\cdot\frac{5}{2}\cdots\frac{n-1}{2}}{1\cdot 2\ldots\frac{n-2}{2}}=(-)^{\frac{1}{2}n}\frac{n\,!}{2^n(\frac{1}{2}n\,!)^2}.$$

8. The formula (5), viz.,

$$\sqrt{\pi}\cdot \mathrm{P}_n=\frac{1}{n\,!}\int_{-\infty}^{\infty}e^{-(1-a^2)t^2}\left(-\frac{d}{da}\right)^n e^{-a^2t^2}dt,$$

admits of the following transformation:

Since
$$a^n\left(\frac{d}{da}\right)^n e^{-a^2t^2}=t^n\left(\frac{d}{dt}\right)^n e^{-a^2t^2},$$

we obtain
$$\sqrt{\pi}\cdot \mathrm{P}_n=\frac{1}{a^n\cdot n\,!}\int_{-\infty}^{\infty}t^n e^{-(1-a^2)t^2}\left(-\frac{d}{dt}\right)^n e^{-a^2t^2}\,dt,$$

which, by integrating by parts n times, is easily seen to give

$$\sqrt{\pi}\cdot \mathrm{P}_n=\frac{1}{a^n\cdot n\,!}\int_{-\infty}^{\infty}e^{-a^2t^2}\left(\frac{d}{dt}\right)^n e^{-(1-a^2)t^2}t^n dt \quad \ldots\ldots\ldots(17).$$

This result may also be obtained as follows: In § 3 it was shown that

$$\sqrt{\pi}\cdot \mathrm{P}_n=\text{coefficient of } h^n \text{ in } \int_{-\infty}^{\infty}e^{-t^2+2hat-h^2}t^n dt$$

$$=\quad ,,\quad ,,\quad \frac{1}{a^n}\int_{-\infty}^{\infty}e^{-t^2+2ha^2t-a^2h^2}t^n dt$$

$$=\quad ,,\quad ,,\quad \frac{1}{a^n}\int_{-\infty}^{\infty}e^{-(1-a^2)t^2-a^2(t-h)^2}t^n dt$$

$$=\quad ,,\quad ,,\quad \frac{1}{a^n}\int_{-\infty}^{\infty}e^{-(1-a^2)(t+h)^2-a^2t^2}(t+h)^n dt$$

$$=\frac{1}{a^n\cdot n\,!}\int_{-\infty}^{\infty}e^{-a^2t^2}\left(\frac{d}{dt}\right)^n e^{-(1-a^2)t^2}t^n dt.$$

9. It is, perhaps, worth remarking that Ivory's formula

$$\mathrm{P}_n=\frac{1}{2^n\cdot n\,!}\left(\frac{d}{da}\right)^n(a^2-1)^n$$

shows at once that the denominators of the coefficients of the powers of a in the expanded value of P_n contain only powers of 2, for, by the differentiation, we obtain as the coefficient of each power of a a product

of n factors; which must be a multiple of $n!$ Thus we can always tabulate *exactly* Laplace's coefficients for arguments which expressed as decimals do not themselves circulate, *i. e.*, $P_n(\alpha)$ is not a circulating decimal unless α is so.

10. We can obtain an expression similar to (5) for the coefficient of h^n in $\dfrac{1}{(1-2h\alpha+h^2)^m}$,

m being any positive quantity integral or fractional, by means of the gamma function formula

$$\frac{1}{a^m} = \frac{1}{\Gamma m}\int_0^\infty e^{-at}\,t^{m-1}\,dt,$$

for $$\frac{1}{(1-2h\alpha+h^2)^m} = \frac{1}{\Gamma m}\int_0^\infty e^{-(1-\alpha^2)t-(h-\alpha)^2 t}\,t^{m-1}\,dt,$$

and the coefficient of $h^n = \dfrac{1}{\Gamma m.n!}\displaystyle\int_0^\infty t^{m+1}e^{-(1-\alpha^2)t}\left(-\frac{d}{d\alpha}\right)^n e^{-\alpha^2 t}\,dt$...(18).

Of course, instead of (2), the integrals

$$\int_0^\infty \sin at^2\,dt = \int_0^\infty \cos at^2\,dt = \frac{\sqrt{\pi}}{2\sqrt{(2a)}}$$

might have been used, but the forms to which they lead present no point of superiority over those involving exponentials.

11. The most natural way of proving the fundamental theorem

$$\int_{-1}^{1} P_m P_n\,d\alpha = 0 \quad \text{or} \quad \frac{2}{2n+1} \qquad \text{.....(19),}$$

according as m and n are unequal or equal, seems to be by examining the coefficient of $h^m k^n$ in the integral

$$\int_{-1}^{1} \frac{d\alpha}{\sqrt{\{(1-2h\alpha+h^2)\,(1-2k\alpha+k^2)\}}} \qquad \text{.....(20),}$$

instead of employing the differential equation satisfied by P_n.

By means of the indefinite integral

$$\int \frac{dx}{\sqrt{(a+bx+cx^2)}} = \frac{1}{\sqrt{c}}\log\{b+2cx+2\sqrt{c}\,.\,\sqrt{(a+bx+cx^2)}\},$$

we find

$$(20) = \frac{1}{2\sqrt{(hk)}}\log\frac{h+k+hk^2+h^2k-4hk-2\sqrt{(hk)}\,(1-h-k+hk)}{h+k+hk^2+h^2k+4hk-2\sqrt{(hk)}\,(1+h+k+hk)}$$

$$= \frac{1}{2\sqrt{(hk)}}\log\frac{\{1+\sqrt{(hk)}\}^2\{\sqrt{h}-\sqrt{k}\}^2}{\{1-\sqrt{(hk)}\}^2\{\sqrt{h}-\sqrt{k}\}^2} = \frac{1}{\sqrt{(hk)}}\log\frac{1+\sqrt{(hk)}}{1-\sqrt{(hk)}},$$

which gives (19) at once. This is the method adopted by Lord Rayleigh, in his memoir in the "Philosophical Transactions" for 1870 (p. 579), for finding the value of

$$\int_0^1 P_m P_n \, da,$$

which is shown to be equal to the coefficient of $h^m k^n$ in

$$\frac{1}{\sqrt{(hk)}} \log \frac{\{1+\sqrt{(hk)}\}\{\sqrt{h}-\sqrt{k}\}}{h\sqrt{(1+k^2)}-k\sqrt{(1+h^2)}} \quad \ldots\ldots\ldots\ldots \quad (21).$$

In the "Proceedings of the Royal Society," t. xxiii. p. 300 (March 4, 1875), Mr. Todhunter has proved that

$$\{2m(2m+1)-(2n-1)2n\}\int_0^1 P_{2m} P_{2n-1}\, da$$
$$= (-)^{m+n} \frac{1.3.5\ldots 2m-1}{2.4.6\ldots 2m} \cdot \frac{1.3.5\ldots 2n-1}{2.4.6\ldots 2n-2},$$

so that we have the striking theorem that, in the expansion of (21), the coefficient of $h^m k^n$ is equal to zero if m and n be unequal and both even or both uneven, is equal to $\frac{1}{2n+1}$ if m and n be equal, and is equal to

$$\frac{(-)^{\frac{1}{2}(m+n+1)}}{m(m+1)-n(n+1)} \cdot \frac{1.3.5\ldots m-1}{2.4.6\ldots m} \cdot \frac{1.3.5\ldots n}{2.4.6\ldots n-1},$$

if m be even and n uneven.

12. In a memoir entitled "On the Equation of Laplace's Functions, &c." ("Philosophical Transactions," 1857, p. 43), the late Professor W. F. Donkin showed that, if $a = \cos\theta$, then

$$P_n = \frac{1}{n!} (\sin\theta)^{-n} \left(\sin\theta \frac{d}{d\theta} \sin\theta\right)^n .1 \quad \ldots\ldots\ldots \quad (22),$$

and in the same memoir there occur (pp. 51 and 53) the two identities

$$\left(\sin\theta \frac{d}{d\theta} \sin\theta\right)^n \left(\tan\frac{\theta}{2}\right)^n = (1.3.5\ldots 2n-1)(\sin\theta)^{2n} \ldots (23),$$

$$(\sin\theta)^{-n} \left(\sin\theta \frac{d}{d\theta} \sin\theta\right)^n \left(\tan\frac{\theta}{2}\right)^i$$
$$= (1-a^2)^{\frac{1}{2}i} a^{n-i} \left(\frac{d}{da}\frac{1}{a}\right)^n a^{n+i} (1+a)^{n-i} \ldots\ldots (24),$$

the latter resulting from the necessary equality of two expressions for the same quantity, the one found by Boole and the other by Donkin. On (24) Donkin remarks: "This equivalence and that expressed by equation (27), art. 14 [viz. (23)], are instances of theorems by no means obvious or easy to verify directly." In a postscript he states, "Mr. Cayley has been kind enough to communicate to me direct verifications of the equation (27), art. 14, and of the identity referred to in

art. 16 [viz. (24)]. Assuming a formula established in Mr. Cayley's paper 'On certain Formulæ for Differentiation, &c.' ['Cambridge and Dublin Journal,' vol. ii., p. 124, equation (2)], the former of the two theorems just mentioned is easily obtained, the latter not without a good deal of trouble." As Professor Cayley's solutions have not, I believe, been published, I conclude these notes with verifications of the three formulæ (22), (23), (24).

13. Putting $\cot\theta = t$, we have

$$-\frac{d}{dt} = \sin^2\theta\,\frac{d}{d\theta},$$

and equation (22) becomes

$$\mathrm{P}_n = \frac{1}{n\,!}(1+t^2)^{\frac{1}{2}(n+1)}\left(-\frac{d}{dt}\right)^n \frac{1}{\sqrt{(1+t^2)}}.$$

This value of P_n can be easily verified, for, since $\alpha = \cos\theta$ and $t = \cot\theta$,

$\mathrm{P}_n =$ coefficient of h^n in $\dfrac{1}{\sqrt{\left\{1-\dfrac{2ht}{\sqrt{(1+t^2)}}+h^2\right\}}}$

$= \quad$ „ $\quad$ „ $\quad \dfrac{\sqrt{(1+t^2)}}{\sqrt{\{1+t^2-2ht\sqrt{(1+t^2)}+h^2(1+t^2)\}}}$

$= (1+t^2)^{\frac{1}{2}n}$ coefficient of h^n in $\dfrac{\sqrt{(1+t^2)}}{\sqrt{(1+t^2-2ht+h^2)}}$

$= (1+t^2)^{\frac{1}{2}(n+1)}$ coefficient of h^n in $\dfrac{1}{\sqrt{\{1+(h-t)^2\}}}$

$= \dfrac{1}{n\,!}(1+t^2)^{\frac{1}{2}(n+1)}\left(-\dfrac{d}{dt}\right)^n \dfrac{1}{\sqrt{(1+t^2)}}.$

14. Putting, as before, $\cot\theta = t$, (23) takes the form

$$\left(-\frac{d}{dt}\right)^n \frac{\{\sqrt{(1+t^2)}-t\}^n}{\sqrt{(1+t^2)}} = \frac{1\cdot3\cdot5\ldots2n-1}{(1+t^2)^{\frac{1}{2}(2n+1)}} \quad \ldots\ldots\ldots\ldots (25),$$

and, to verify this, we notice that, by Lagrange's theorem,

if $\qquad x = t + hfx,$

then $\dfrac{1}{n\,!}\left(\dfrac{d}{dt}\right)^n (f^n t\,.\,\mathrm{F}'t) =$ coefficient of h^n in the expansion of $\mathrm{F}'x\,.\,\dfrac{dx}{dt}$.

In our case, $\qquad x = t + h\,\{\sqrt{(1+x^2)} - x\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (26),$

and $\qquad \mathrm{F}'x = \dfrac{1}{\sqrt{(1+x^2)}}\,;$

therefore $\mathrm{F}'x\,.\,\dfrac{dx}{dt} = \dfrac{1}{\sqrt{(1+x^2)}-h\,\{x-\sqrt{(1+x^2)}\}} = \dfrac{1}{\sqrt{(1+x^2)}+x-t}.$

Writing, for the moment, $\sqrt{(1+x^2)}+x = p$, we have

$$\frac{1}{p} = \sqrt{(1+x^2)} - x,$$

whence $$1+x^2 = \left(x+\frac{1}{p}\right)^2 \text{ and } 1+2px-p^2 = 0.$$

Thus $x = \dfrac{p^2-1}{2p}$, and (26) becomes

$$\frac{p^2-1}{2p} = t+\frac{h}{p},$$

viz., $$(p-t)^2 = t^2+2h+1,$$

so that $$\frac{1}{\sqrt{(1+x^2)}+x-t} = \frac{1}{p-t} = \frac{1}{\sqrt{1+2h+t^2}},$$

in the expansion of which quantity the coefficient of h^n is

$$\frac{1}{n!}\cdot(-)^n \frac{1.3.5\ ..\ 2n-1}{(1+t^2)^{\frac{1}{2}(2n+1)}}$$

thus proving (25).

15. To verify (24), we observe that

$$\frac{1}{n!}(\sin\theta)^{-n-1}\left(\sin^2\theta\frac{d}{d\theta}\right)^n \sin\theta\left(\tan\frac{\theta}{2}\right)^i$$

$$= \frac{1}{n!}(1+t^2)^{\frac{1}{2}(n+1)}\left(-\frac{d}{dt}\right)^n \frac{\{\sqrt{(1+t^2)}-t\}^i}{\sqrt{(1+t^2)}}$$

$$= (1+t^2)^{\frac{1}{2}(n+1)} \text{ coefficient of } h^n \text{ in } \frac{\{\sqrt{(1+t^2-2th+h^2)}-t+h\}^i}{\sqrt{(1+t^2-2th+h^2)}}$$

$$= (1+t^2)^{\frac{1}{2}i} \text{ coefficient of } h^n \text{ in}$$

$$\frac{\{\sqrt{[1+t^2-2t\sqrt{(1+t^2)}\,h+h^2(1+t^2)]}-t+h\sqrt{(1+t^2)}\}^i}{\sqrt{\{1+t^2-2t\sqrt{(1+t^2)}h+h^2(1+t^2)\}}}$$

$$= \text{coefficient of } h^n \text{ in } (\operatorname{cosec}\theta)^i \frac{\{\sqrt{(1-2h\cos\theta+h^2)}-\cos\theta+h\}^i}{\sqrt{(1-2h\cos\theta+h^2)}} \quad \ldots(27).$$

Now, the right-hand side of (24) is

$$\frac{(1-a^2)^{\frac{1}{2}i}}{a^{i-1}}\cdot a^n\left(\frac{d}{a\,da}\right)^n \frac{1}{a}\cdot a^n(1+a)^n\cdot\left(\frac{a}{1+a}\right)^i;$$

and, taking $a^2=\beta$,

$$\frac{a^n}{n!}\left(\frac{d}{a\,da}\right)^n\left\{a^n(1+a)^n\cdot\frac{1}{a}\left(\frac{a}{1+a}\right)^i\right\}$$

$$= \frac{\beta^{\frac{1}{2}n}}{n!}2^n\left(\frac{d}{d\beta}\right)^n\left\{(\sqrt{\beta}+\beta)^n\cdot\frac{1}{\sqrt{\beta}}\cdot\left(\frac{\sqrt{\beta}}{1+\sqrt{\beta}}\right)^i\right\}$$

$$= \text{coefficient of } h^n \text{ in } \frac{1}{\sqrt{x}}\left(\frac{\sqrt{x}}{1+\sqrt{x}}\right)^i\frac{dx}{d\beta},$$

where $$x = \beta+2h\sqrt{\beta}\,(\sqrt{x}+x) \quad \ldots\ldots\ldots\ldots\ldots\ldots (28);$$

and, in the formation of $\dfrac{dx}{d\beta}$, the $\sqrt{\beta}$ that multiplies h is to be treated as

constant. Writing for β its value, viz. $\cos^2\theta$, (28) gives

$$x(1-2h\cos\theta)-2h\cos\theta\sqrt{x}=\cos^2\theta;$$

whence
$$\sqrt{x}=\frac{h+\sqrt{(1-2h\cos\theta+h^2)}}{1-2h\cos\theta}\cos\theta,$$

and we find that

$$\frac{\sqrt{x}}{1+\sqrt{x}}=\frac{\sqrt{(1-2h\cos\theta+h^2)}-\cos\theta+h}{\sin^2\theta}\cos\theta;$$

also, on the understanding mentioned above, with regard to the factor $\sqrt{\beta}$,

$$\frac{1}{\sqrt{x}}\frac{dx}{d\beta}=\frac{1}{\cos\theta\sqrt{(1-2h\cos\theta+h^2)}};$$

so that the coefficient of h^n in $\dfrac{1}{\sqrt{x}}\left(\dfrac{\sqrt{x}}{1+\sqrt{x}}\right)^i\dfrac{dx}{d\beta}$

$=$ coefficient of h^n in $\dfrac{\{\sqrt{(1-2h\cos\theta+h^2)}-\cos\theta+h\}^i}{(\sin\theta)^{2i}\sqrt{(1-2h\cos\theta+h^2)}}(\cos\theta)^{i-1}$;

and this, on multiplication by

$$\frac{(1-a^2)^{\frac{1}{2}i}}{a^{i-1}},\ \text{that is, by}\ \frac{(\sin\theta)^i}{(\cos\theta)^{i-1}},$$

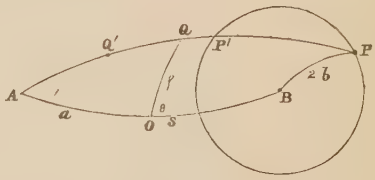
becomes identical with (27).

On the Mechanical Description of a Sphero-Conic. *By* Mr. Hart.

[*Read May 13th*, 1875.]

Let P be any point on a small circle of a sphere whose centre is B and radius subtends angle $2b$ at centre of sphere.

Let A be a fixed point, $AB=2a$, O the middle point of AB.



Then the locus of Q, the middle point of AP, is a sphero-conic. For if $OQ=\rho$, $QOB=\theta$, $AQ=QP=x$,

$$\cos x=\cos a\cos\rho-\sin a\sin\rho\cos\theta \quad \ldots\ldots\ldots\ldots (1),$$

$$\frac{\cos 2b-\cos 2a\cos 2x}{\sin 2a\sin 2x}=\frac{\cos\rho-\cos a\cos x}{\sin a\sin x} \quad \ldots\ldots\ldots\ldots (2);$$

whence, eliminating x,

$$\tan^2\rho=\frac{\sin^2 b}{(\cos^2 a-\sin^2 b)\sin^2\theta+\cos^2 b\cos^2\theta},$$

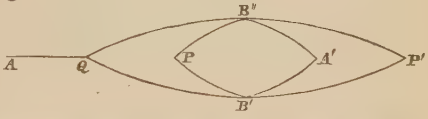
which is the polar equation to a sphero-conic whose centre is O, minor axis coincides with AB and $=2b$, major axis perpendicular to AB

$= 2\sin^{-1}\left(\frac{\sin b}{\cos a}\right)$; whence we see that A, B are poles with respect to the sphere of the cyclic arcs of the conic.

Also, since $AQ = \frac{1}{2}AP$, $AQ' = \frac{1}{2}AP'$, we see that—If through the pole of a cyclic arc an arc be drawn cutting a conic, the product of the trigonometrical tangents of the arcs is constant.

If now we cause a point P to describe a small circle on a sphere by means of a link having the same curvature as a great circle, we only require a linkage to enable us to bisect any arc AP.

Such a linkage is the following:—



QB″P′, QB′P′ are two curved links = 180°; B″P, PB′, B′A′, A′B″ are four equal links, = a, a constant. It is evident that Q, P, A′, P′ are four points on the same great circle, and QP′ = 180°.*

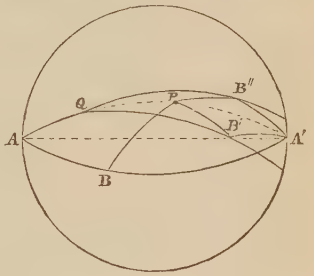
If, then, A be a point on the sphere, and A′ its antipode,

$$AA' = 180° = QP',$$

or

$$AQ = A'P'$$
$$= QP,$$

or AP is bisected in Q.



The application of this to the description of a sphero-conic is evident.

A Parallel Motion. *By* Mr. Hart.

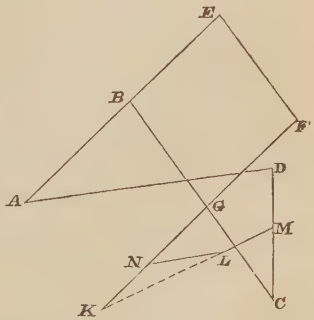
[*Read May 13th*, 1875.]

I. ABCD is a skew parallelogram, and the links in the figure satisfy the following conditions:—

$$AB = CD = 2CM = 2r,$$
$$EB = BG = GC = \tfrac{1}{2}AD = p,$$
$$EF = FG = a,$$
$$NG = NK = NL = \frac{qr}{2a},$$

(where $q = r + p$,)

$$LM = r - p;$$



then, if $rq^2 = 4a^2p$,

K always lies in LM produced,

G „ LG perpendicular to LM.

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* By employing a linkage ABCD analogous to that given in the *Messenger*, Oct. 1874, it is easily seen that only five links are requisite for the description of a sphero-conic.

To prove this, produce BF, GM to meet in Z, then it is easily seen that the triangles GFZ, GMK are similar, and hence

$$\angle BFG = \angle GMK.$$

Also, if BF, GE intersect at right angles in O, it may be shown that

$$\frac{FO}{FG} = \frac{ML}{MG}.$$

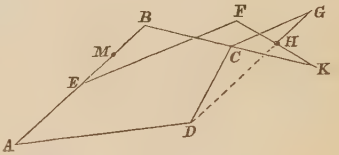
Consequently, GL is always perpendicular to LM, and since GLK is a right angle, L is always at the same distance from N, viz. NL = NG = NK.

If the links KNGF, NL, LM be removed, and the points G, F be fixed, then the remaining *five* links serve to describe the Limaçon, the point M being the tracing point.

II. $AB = AD = a$,

$BC = CD = CK = \frac{1}{2}EF = b$,

$EM = MB = HG = HF = HK = \frac{b^2}{a}$;



then D lies in HG produced.

Since $BM : BC :: BC : BA$,

$$\angle BAC = \angle BCM = \tfrac{1}{2}\angle GCK;$$

therefore $\angle BAD = \angle GCK.$

For a similar reason $\angle BCD = \angle GHK;$

therefore
$$\angle CGH = \tfrac{1}{2}\angle BCD - \tfrac{1}{2}\angle BAD$$
$$= \tfrac{1}{2}\{\pi - DCK - GCK\}$$
$$= \angle CGD; \textit{ i.e.}, \text{ H lies on GD.}$$

If DC be produced to D′, where CD′ = CD; then D′ lies on GD′ perpendicular to GH. If the links DC, CG, GH be removed, but D, C fixed, then the remaining five links serve to describe the Limaçon, H being the tracing point.

These two parallel motions are particular cases of a more general but more complicated one. I found the condition that the point in Mr. S. Roberts' three-bar motion should describe the inverse of a Limaçon, and then made the points of inversion the two fixed points. The general condition

$$\left\{az - \frac{(r^2-a^2)(r+p)}{r}\right\}^2 = 4rp(r^2-a^2)$$

becomes then $r(r-p)^2 = -4pa^2$ ..... (1),

or $rq^2 = -4pa^2$;

whence we see p is negative (see fig.), and

$$rq = a^2 \text{ ..... (2).}$$

In the same way I deduced Mr. Kempe's motion, not knowing of its existence, from the other three-bar motion, which, owing to its symmetry, only gives one result; the general condition I gave (*Messenger*, Dec. 1874)

$$\{a^2m+b^2c\}^2 = b^2(a^2-b^2)(a^2-c^2);$$

putting $c+2m = \pm a$, $\quad 2b^2 = \pm a\{\pm a - c\}$.

June 10th, 1875.

Prof. H. J. S. SMITH, F.R.S., President, in the Chair.

Messrs. Hunter, Rhodes, and Wright were elected Members; Mr. W. Tanner, B.A., Jesus College, Oxford, was proposed for election; and Mr. J. H. Röhrs was admitted into the Society.

Communications were made by Prof. Cayley, "On some Figures of Curves in Three-bar Motion; Dr. Sylvester, "On James Watt's Parallel Motion"; and Mr. T. Cotterill, "On the Correspondence of Points Collinear with a Fixed Origin." The Chairman and Dr. Hirst spoke on the subject of the last communication.

The following presents were received:—

"Bulletin de la Société Mathématique de France," tome iii., Avril, No. 1.

"Jahrbuch über die Fortschritte der Mathematik" (Ohrtmann, Müller), Fünfter Band, Jahrgang 1873, Heft i.

"Bulletin des Sciences Mathématiques et Physiques," Fev. 1875.

"Mémoires de la Société des Sciences de Bordeaux," tome x., 2e cahier. Paris, 1875.

C. G. Reuschle, "Tafeln complexer Primzahler welche aus Wurzeln der Einheit gebildet sind." Berlin, 1875.

"Sur la résolution des équations du 2me, 3me et 4me degré par la fonction $\overset{n}{r}(X)$," par Dr. Axel S. Guldberg (20 Dec. 1872).

"Beretning om der internationale Meter-Kommissions Mφde i Paris, 24 Sep.—12 Octbr. 1872," af Dr. O. J. Broch. Christiania, 1874.

"Partielle Differential-Gleichungen 1, 0, in denen die unbekannte Funktion explicite vorkommt von Sophus Lie" (besonderer Abdruck nach den Verhandlungen der Gesellschaft d. W. zu Christiania, 1873).

"Neue Integrations-Methode eines $2n$-gliedrigen Pfaffschen Problems," von Sophus Lie (vorgelegt im October 1873).

"Ueber partielle Differential-Gleichungen 1, 0," von Sophus Lie.

"Zur analytischen Theorie der Berührungs-Transformationen," von Sophus Lie.

"Bidrag til Theorien for Dissociationen," af C. M. Guldberg (10 Mai 1872).

"Bemærkninger om Formelen for Hφidemaaling med Barometer," af C. M. Guldberg.

The last eight pamphlets presented by the Royal University of Norway, Christiania.

On the Integration of Discontinuous Functions.

By Henry J. Stephen Smith, F.R.S.

1. Riemann, in his Memoir "Ueber die Darstellbarkeit einer Function durch eine Trigonometrische Reihe" (Abhandlungen der k. Gesellschaft der Wissenschaften zu Göttingen, vol. xiii., p. 87), has given an important theorem which serves to determine whether a function $f(x)$ which is discontinuous, but not infinite, between the finite limits a and b, does or does not admit of integration between those limits, the variable x, as well as the limits a and b, being supposed real. Some further discussion of this theorem would seem to be desirable, partly because, in one particular at least, Riemann's demonstration is wanting in formal accuracy, and partly because the theorem itself appears to have been misunderstood, and to have been made the basis of erroneous inferences.

2. Let d be any given positive quantity, and let the interval $b-a$ be divided into any *segments* whatever, $\delta_1 = x_1 - a$, $\delta_2 = x_2 - x_1$, ..., $\delta_n = b - x_{n-1}$, subject only to the condition that none of these segments surpasses d. We may term d the *norm* of the division; it is evident that there is an infinite number of different divisions having a given norm; and that a division appertaining to any given norm, appertains also to every greater norm. Let $\epsilon_1, \epsilon_2, \ldots\ldots \epsilon_n$ be positive proper fractions; if, when the norm d is diminished indefinitely, the sum

$$S = \delta_1 f(a+\epsilon_1\delta_1) + \delta_2 f(x_1+\epsilon_2\delta_2) + \ldots\ldots + \delta_n f(x_{n-1}+\epsilon_n\delta_n)$$

converges to a definite limit, whatever be the mode of division, and whatever be the fractions $\epsilon_1, \epsilon_2, \ldots\ldots \epsilon_n$, that limit is represented by the symbol $\int_a^b f(x)\,dx$, and the function $f(x)$ is said to admit of integration between the limits a and b. We shall call the values of $f(x)$ corresponding to the points of any segment the *ordinates* of that segment; by the *ordinate difference* of a segment we shall understand the difference between the greatest and least ordinates of the segment.

For any given division $\delta_1, \delta_2, \ldots\ldots \delta_n$, the greatest value of S is obtained by taking the maximum ordinate of each segment, and the least value of S by taking the minimum ordinate of each segment; if D_i is the ordinate difference of the segment δ_i, the difference θ between these two values of S is $\theta = \delta_1 D_1 + \delta_2 D_2 + \ldots\ldots + \delta_n D_n$.

But, for a given norm d, the greatest value of S, and the least value of S, will in general result, not from one and the same division, but from two different divisions, each of them having the given norm. Hence the difference Θ between the greatest and the least values that S can acquire for a given norm, is, in general, greater than the greatest of the differences θ. To satisfy ourselves, in any given case, that S converges to a definite limit, when d is diminished without limit, we must be sure that Θ diminishes without limit; and it is not enough to show (as the form of Riemann's proof would seem to imply) that θ diminishes without limit, even if this should be shown for every division having the norm d.

3. Let $A(d)$ be the greatest value of S appertaining to a given norm d, and let $B(d)$ be the least value of S appertaining to the same norm. If d_1 and d_2 are any two norms, of which d_1 is greater than d_2, it is evident that $A(d_1) \geqq A(d_2)$, $B(d_1) \leqq B(d_2)$, because every division appertaining to the norm d_2 also appertains to the norm d_1. And it may be proved (although, for brevity, we omit the demonstration here) that, given any norm d_1, we can always assign a norm d_2, less than d_1, which shall satisfy the inequalities $A(d_1) > A(d_2)$, $B(d_1) < B(d_2)$; except only when the function is such that the maximum (or minimum) ordinate is the same, throughout the whole interval, for all segments however small. In this excepted case, which is one by no means inconceivable, the value of $A(d)$, [or of $B(d)$,] is independent of d, and is simply $h(b-a)$, where h is the maximum (or minimum) ordinate common to all segments of the interval $b-a$. In all other cases, it is possible to assign a series of norms, decreasing without limit, and such that the corresponding maximum values of S form a decreasing series, while the corresponding minimum values of S form an increasing series.

Besides the maximum and minimum values of S corresponding to a given norm, we have also to consider the maximum and minimum values of S corresponding to a given division. Let $P(d)$ be the maximum value of S appertaining to a given division of norm d, and let $Q(d')$ be the minimum value of S appertaining to a different division, having the same norm or a different norm. It is important to observe that we shall always have $P(d) > Q(d')$, the sign of equality being inadmissible, except when the function is such as to be represented geometrically by

a single segment, or a system of segments, parallel to the axis of x. Leaving out of consideration the excepted case, we may enunciate the theorem—"The least value of S that can be obtained by taking, in any division whatever, the greatest ordinate of each segment, is greater than the greatest value that can be obtained by taking, in any division whatever, the least ordinate of each segment." To prove this theorem, let the two divisions, which give the values $\mathrm{P}(d)$ and $\mathrm{Q}(d')$, be simultaneously applied to the interval $b-a$. To obtain $\mathrm{P}(d)$, each segment in the resulting division will have to be multiplied by its greatest ordinate, or by a still greater ordinate in some adjacent segment; whereas to obtain $\mathrm{Q}(d')$ each segment will have to be multiplied by its least ordinate, or by a still less ordinate. It follows that we have, in general, $\mathrm{P}(d) > \mathrm{Q}(d')$. If, however, we regard the interval $b-a$ as composed of segments $l_1, l_2, \ldots$, each of which has for its extremities points which are also extremities of segments in each of the two given divisions, we shall find that the inequality $\mathrm{P}(d) > \mathrm{Q}(d')$ must be replaced by the equality $\mathrm{P}(d) = \mathrm{Q}(d')$, if it should so happen that the maximum ordinate of each segment l is the same as its minimum ordinate; *i.e.*, if the function $f(x)$ is represented geometrically by a series of segments parallel to the axis of x, and respectively equal to the segments $l_1, l_2, \ldots$

4. Again, let $\mathrm{B}'(d)$ be the least value of S corresponding to the division which gives $\mathrm{A}(d)$; and let $\mathrm{A}'(d)$ be the greatest value of S corresponding to the division which gives $\mathrm{B}(d)$; it is evident from what has been said that we shall have the inequalities

$$\mathrm{A}(d) > \mathrm{A}'(d) > \mathrm{B}'(d) > \mathrm{B}(d).$$

Now

$$\mathrm{A}(d) - \mathrm{B}(d) = [\mathrm{A}(d) - \mathrm{B}'(d)] + [\mathrm{A}'(d) - \mathrm{B}(d)] - [\mathrm{A}'(d) - \mathrm{B}'(d)];$$

and

$$\mathrm{A}'(d) \geqq \mathrm{B}'(d);$$

therefore

$$\mathrm{A}(d) - \mathrm{B}(d) \leqq [\mathrm{A}(d) - \mathrm{B}'(d)] + [\mathrm{A}'(d) - \mathrm{B}(d)].$$

Hence, to prove the evanescence of $\mathrm{A}(d) - \mathrm{B}(d)$ or Θ, it suffices to prove the evanescence of $\mathrm{A}(d) - \mathrm{B}'(d)$, and of $\mathrm{A}'(d) - \mathrm{B}(d)$, which are, in fact, the two values of θ corresponding to the two divisions which give the absolutely greatest and least values of S for the norm d.

5. The theorem of Riemann may be enunciated as follows:—

"Let σ be any given quantity, however small; if, in every division of norm d, the sum of the segments, of which the ordinate differences surpass σ, diminishes without limit, as d diminishes without limit, the function admits of integration; and, *vice versâ*, if the function admits of integration, the sum of these segments diminishes without limit with d."

The following (with a slight modification suggested by the preceding considerations) is Riemann's demonstration of the first part of the theorem:

Let s_1 be the sum of the segments which, in the division corresponding to $A(d)$ and $B'(d)$, have ordinate differences surpassing σ; and let Ω be the greatest ordinate difference in any division appertaining to the norm d; Ω is necessarily finite, because all the ordinates are finite. The contribution of the segments s_1 to the difference $A(d)-B'(d)$ cannot surpass $s_1 \times \Omega$, and the contribution of the remaining segments cannot surpass $\sigma \times (b-a-s_1)$; *i.e.*,

$$A(d)-B'(d) \leqq s_1 \times \Omega + \sigma (b-a-s_1).$$

Similarly, if s_2 is the sum of the segments which, in the division corresponding to $A'(d)$ and $B(d)$, have ordinate differences surpassing σ,

$$A'(d)-B(d) \leqq s_2 \times \Omega + \sigma (b-a-s_2).$$

Adding these two inequalities, we find

$$A(d)-B(d) \leqq (s_1+s_2)(\Omega-\sigma)+2\sigma(b-a).$$

But σ may be taken as small as we please, and, by hypothesis, however small σ may be, d can always be taken so small as to render s_1 and s_2 as small as we please; *i.e.*, the difference $A(d)-B(d) = \Theta$ diminishes without limit with d, and $f(x)$ admits of integration between the limits a and b.

6. Riemann's demonstration of the second part of the theorem requires no modification. For, if S converges to a definite limit, Θ must be comminuent with d, and, *à fortiori*, each of the quantities θ must be comminuent with d. But, evidently, in any given division in which s is the sum of the segments having ordinate differences which surpass σ, $\sigma s \leqq \theta$. Hence, however small the given quantity σ may be, we can always, by taking d small enough, make $\frac{\theta}{\sigma}$ less than any assigned quantity; *i.e.*, if S converges to a definite limit, s must diminish without limit at the same time with d.

7. It will be observed that, in order to establish the convergence of S to a definite limit, it is sufficient to know that the sum of the segments, having ordinate differences surpassing σ, is comminuent with d in each of two specified divisions [viz., in the division which gives $A(d)$ the maximum value of S, and in that which gives $B(d)$ the minimum value of S]. Hence, if these two sums are comminuent with d, the corresponding sum in any other division of norm d is also comminuent with d.

8. Let us suppose that the function $f(x)$ has any number of dis-

continuities between a and b; and let there be $\psi(\sigma)$ points at which there are discontinuities surpassing σ. (We say that a discontinuity surpassing σ exists at a given point, when any segment, however small, being taken which includes that point, the ordinate difference of the segment surpasses σ.) If $\psi(\sigma)$ has a finite and assignable value for every value of σ, however small, the condition of integrability is certainly satisfied, even if $\psi(\sigma)$ increase without limit, when σ diminishes without limit. For, in any division of norm d, the sum of the segments having ordinate differences which surpass σ, cannot surpass $2d \times \psi(\sigma)$; and, however small σ may be, d can be taken so small that $2d \times \psi(\sigma)$ shall be less than any quantity that can be assigned. As an example, we may take the function considered by Riemann, viz.,

$$f(x) = \frac{(x)}{1} + \frac{(2x)}{4} + \frac{(3x)}{9} + \ldots\ldots,$$

where, by (x) we are to understand the (positive or negative) excess of x above the whole number nearest to x; or, if x lies half-way between two whole numbers, the arithmetical mean between the two differences $\frac{1}{2}$ and $-\frac{1}{2}$, *i. e.*, zero. In this function, if $x = \frac{m}{2n}$, where m and $2n$ are relatively prime, we have

$$f\left(\frac{m}{2n} + 0\right) = f\left(\frac{m}{2n}\right) - \frac{\pi^2}{16n^2},$$

$$f\left(\frac{m}{2} - 0\right) = f\left(\frac{m}{2n}\right) + \frac{\pi^2}{16n^2}.$$

Thus the number of discontinuities in any given interval is infinitely great. But the number of discontinuities which in any given interval surpass a given quantity σ, is always finite. For example, the number of discontinuities between 0 and 1 which surpass σ, is equal to the number of irreducible proper fractions, having even denominators $2n$, which verify the inequality $\frac{\pi^2}{8n^2} > \sigma$; or, if $\varphi(m)$ be the number of numbers not surpassing m and prime to m, and if h be the greatest integer not surpassing $\frac{\pi}{2\sqrt{2\sigma}}$, the number of discontinuities in question is

$$\varphi(1) + \varphi(2) + \ldots\ldots \varphi(h) = \psi(\sigma),$$

which is evidently finite for any given value of σ, although it increases without limit when σ diminishes without limit.

9. Next, let us suppose that $f(x)$ in the interval $b-a$ has an infinite number of discontinuities surpassing a given quantity σ. The points at which these discontinuities occur may either "completely fill" one or more finite portions of the interval $b-a$, or there may be no finite

portion of that interval which is "completely filled" by them. A system of points is said to "fill completely" a given interval when, any segment of the interval being taken, however small, one point at least of the system lies on that segment. Thus the *rational* points on any line, *i. e.*, the points of which the abscissæ are rational, completely fill any segment whatever upon the line. We may observe that the assertion, that any given segment of an interval contains at least one point of a given system, is equivalent to the assertion that any given segment contains an infinite number [*i. e.*, a number greater than any that can be assigned] of the points of the system. For we may divide the given segment into as many parts as we please, and each of them must contain at least one point of the system.

10. When the points at which there occur discontinuities surpassing σ completely fill any finite portion of the interval $b-a$, the function $f(x)$ is certainly incapable of integration. For, if l be the total length of the segments which are completely filled, we have evidently $\theta > \sigma l$ for any division of any norm d; *i. e.*, it is impossible that Θ should diminish without limit with d.

But points may exist in an infinite number within a finite interval, without completely filling any portion of that interval. Whenever this happens, it must be possible in any given segment of the interval, however small, to take a finite part such that it shall contain no point of the system; otherwise, the segment in question would be completely filled. We give a few examples of such systems of points, the limits of the interval being in each case 0 and 1. We shall say, for brevity, that points are in close order on any segment when they completely fill it, and in loose order when they do not completely fill it, or any part of it however small.

11. (i.) Let the system of points be defined by the equation $x = \frac{1}{a}$, a being any positive integer. It will be seen, (1) that these points are infinite in number; (2) that they are indefinitely condensed in the vicinity of the origin; (3) that they are in loose order over the whole interval, no segment, even in the immediate vicinity of the origin, being completely filled. For if d be any given quantity, however small, we can always find a finite integral number such that $\frac{1}{m} < d$, and then the finite spaces $\left(\frac{1}{m+1}, \frac{1}{m}\right)$, $\left(\frac{1}{m+2}, \frac{1}{m+1}\right)$, &c. ..... all lie on the segment $(0, d)$, and are all free from points of the system, if we leave their initial and terminal points out of account.

12. (ii.) Let the system of points be defined by the equation $x = \frac{1}{a_1} + \frac{1}{a_2}$, where a_1 and a_2 are any positive integers. Here, it is evi-

dent that the points are indefinitely condensed in the vicinity of each of the points of the system (i). But it can also be shown that they are in loose order over the whole interval from 0 to 1. Let $x = L_1$, $x = L_2$, $(L_1 < L_2,)$ be two consecutive points of the system (i); let μ be any positive quantity whatever, and consider the segment $\left(\frac{\mu L_1 + L_2}{\mu + 1}, L_2\right)$. If $x = \frac{1}{a_1} + \frac{1}{a_2}$ lies on this segment, we must have $\frac{1}{a_1} \leq L_1$, $\frac{1}{a_2} \leq L_1$, because no point of the system (i) lies on the interval (L_1, L_2); and also $\frac{1}{a_1} + \frac{1}{a_2} \geq \frac{\mu L_1 + L_2}{\mu + 1}$; whence $a_1 \leq \frac{\mu+1}{L_2 - L_1}$, $a_2 \leq \frac{\mu+1}{L_2 - L_1}$. These inequalities show that, if, from the beginning of any free segment in the system (i), we cut off as small a part as we please (which we may do by taking μ great enough), the remaining portion of that segment will contain only a finite number of points belonging to the system (ii). And this suffices to prove that the points of the system are in loose order; for if d be any segment, however small, situated anywhere in the interval from 0 to 1, we can certainly find on this segment a part free from points of the system (i), and, by what has just been proved, parts of that part will be free from points of the system (ii).

13. (iii.) Let a system of points P_{s+1} be defined by the equation $x = \frac{1}{a_1} + \frac{1}{a_2} + \ldots \frac{1}{a_{s+1}}$, where $a_1, a_2, \ldots a_{s+1}$ are positive integers. Assuming (what has just been proved for $s=2$) that the system P_s is in loose order over the whole interval from 0 to 1, we shall prove the same thing for the system P_{s+1}. Let $x = L_1$, $x = L_2$ be any two consecutive points of the system P_s; and consider as before the interval $\left(\frac{\mu L_1 + L_2}{\mu + 1}, L_2\right)$. If the point P_{s+1}, or $x = \frac{1}{a_1} + \frac{1}{a_2} + \ldots \frac{1}{a_{s+1}}$, lies on this interval, we must have, besides the inequality

$$\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_{s+1}} > \frac{\mu L_1 + L_2}{\mu + 1},$$

the $s+1$ inequalities included in the formula

$$\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_{s+1}} \leq L_1 + \frac{1}{a_i},$$

because no point of the system P_s can be between L_1 and L_2. These inequalities give $a_i \leq \frac{\mu+1}{L_2 - L_1}$, $i = 1, 2, 3 \ldots s+1$,

whence we may infer, precisely as in the case in which $s=2$, that the points P_{s+1} are in loose order over the whole of interval from 0 to 1.

14. Let $f(x)$ be a function, which coincides with a given continuous function $\phi(x)$ for all values of x between 0 and 1, except at the points P_{s+1}; and let the difference between $f(x)$ and $\phi(x)$ at those points not

exceed the finite quantity σ. It may be shown that $f(x)$ is integrable between the limits 0 and 1, and that

$$\int_0^1 f(x)\,dx = \int_0^1 \phi(x)\,dx.$$

For, take any small interval from 0 to δ; the points P_1 which lie outside it, between δ to 1, are finite in number and at finite distances from one another. Let there be Δ_1 of them; from each of them measure a space δ_1 to the right; the number of points P_2, lying outside of the measured spaces $\delta + \Delta_1\delta_1$, is necessarily finite; and these points are at finite distances from one another. Let their number be Δ_2, and measure a distance δ_3 to the right of each of them. Proceeding in this way, we shall obtain measured spaces amounting in all to

$$\delta + \Delta_1\delta_1 + \Delta_2\delta_2 + \ldots\ldots + \Delta_{s+1}\delta_{s+1} = \mathrm{H}.$$

Let ϵ be any given quantity however small; and in the preceding construction let

$$\delta < \frac{\epsilon}{3(s+2)},\ \delta_1 < \frac{\epsilon}{3(s+2)\Delta_1},\ \delta_2 < \frac{\epsilon}{3(s+2)\Delta_2},\ \ldots\ldots\ \delta_{s+1} < \frac{\epsilon}{3(s+2)\Delta_{s+1}};$$

we shall thus have $\mathrm{H} < \frac{1}{3}\epsilon$. Let d be the least of the spaces $\delta, \delta_1, \delta_2 \ldots \delta_{s+1}$; it may be shown that, in any division of norm d, the sum of the segments containing points P_{s+1} cannot exceed 3H. For all the points P_{s+1} lie on the measured spaces; and supposing (which is the most unfavourable case) that one of those spaces begins and ends with a point P_{s+1}, we can at most triple it, by imagining a segment equal to d placed on each side of it. Thus, in every division of norm d, the sum of the segments containing the points of discontinuity is less than ϵ; whence we infer, by Riemann's theorem, that $\int_0^1 f(x)\,dx$ has the same value as $\int_0^1 \phi(x)\,dx$.

15. (iv.) Let m be any given integral number greater than 2. Divide the interval from 0 to 1 into m equal parts; and exempt the last segment from any subsequent division. Divide each of the remaining $m-1$ segments into m equal parts; and exempt the last segment of each from any subsequent division. If this operation be continued *ad infinitum*, we shall obtain an infinite number of points of division P upon the line from 0 to 1. These points are in loose order: for if d be any segment however small, situated anywhere in the interval from 0 to 1, we may take an index k which satisfies the inequality $\frac{1}{m} < \frac{1}{2}d$; and then determine a segment of the type $\left(\frac{a}{m^k}, \frac{a+1}{m^k}\right)$ lying entirely on the segment d. But this segment is either itself an exempted segment or its m^{th} part is so. It will be seen that, after k operations, the

sum of the exempted segments amounts to $1-\left(1-\frac{1}{m}\right)^k$; so that, as k increases without limit, the points of division P occur upon segments which occupy only an infinitesimal portion of the interval from 0 to 1. And it may be inferred that a function, having any finite discontinuities at the points P, would be integrable. For, if d be any given small quantity, let the index k be determined by the inequalities $\frac{1}{m^k} > d > \frac{1}{m^{k+1}}$; the number N of excepted segments which surpass $\frac{1}{m^k}$ is

$$1+(m-1)+(m-1)^2+\ldots\ldots+(m-1)^{k-2};$$

and the sum of the remaining segments is

$$\left(1-\frac{1}{m}\right)^{k-1}.$$

It is evident that in any division of norm d, the sum of the segments containing points P cannot exceed

$$\left(1-\frac{1}{m}\right)^{k-1}+2\mathrm{N}d.$$

But, as d decreases, and k increases, without limit, $\left(1-\frac{1}{m}\right)^{k-1}$ and $2\mathrm{N}d$, which is less than $\frac{2\mathrm{N}}{m^k}$, both decrease without limit; *i.e.*, in any division of norm d, the sum of the segments containing points of discontinuity diminishes without limit with d; and the function is integrable.

16. (v.) Let us now, as in the last example, divide the interval from 0 to 1 into m equal parts, exempting the last segment from any further division; let us divide each of the remaining $m-1$ segments by m^2, exempting the last segment of each segment; let us again divide each of the remaining $(m-1)(m^2-1)$ segments by m^3, exempting the last segment of each segment; and so on continually. After $k-1$ operations we shall have

$$\mathrm{N}=1+(m-1)+(m-1)(m^2-1)+\ \ldots\ +(m-1)(m^2-1)\ldots(m^{k-2}-1)$$

exempted segments, of which the sum will be

$$1-\left(1-\frac{1}{m}\right)\left(1-\frac{1}{m^2}\right)\ldots\left(1-\frac{1}{m^{k-1}}\right).$$

This sum, when k is increased without limit, approximates to the finite limit $1-\mathrm{E}\left(\frac{1}{m}\right)$; where $\mathrm{E}\left(\frac{1}{m}\right)$ is the Eulerian product $\prod_1^\infty\left(1-\frac{1}{m^k}\right)$, and is certainly different from zero. The points of division Q exist in loose order over the whole interval. For, if d be any small segment of that interval, and if $\frac{1}{m^{\frac{1}{2}k(k-1)}} < \frac{1}{2}d$, a segment of the type $\left(\frac{a}{m^{\frac{1}{2}k(k-1)}}, \frac{a+1}{m^{\frac{1}{2}k(k-1)}}\right)$ can be found lying entirely on the segment d, and this segment is either

itself exempted, or its $\left(\frac{1}{m^k}\right)^{\text{th}}$ part is exempted. But a function having finite discontinuities at the points Q would be incapable of integration. For, if d be any norm, and $\delta < \frac{1}{m^{\frac{1}{2}k(k-1)}} < d$, in the division

$$\delta + \frac{i}{m^{\frac{1}{2}k(k-1)}}, \quad i = 0, 1, 2, 3, \ldots\ldots$$

(which is a division of norm d), the sum of the segments containing points of discontinuity is

$$\left(1-\frac{1}{m}\right)\left(1-\frac{1}{m^2}\right)\ldots\ldots\left(1-\frac{1}{m^{k-1}}\right)+\frac{N}{m^{\frac{1}{2}k(k-1)}},$$

which approximates to the finite limit $E\left(\frac{1}{m}\right)$ when d is diminished, and k is increased without limit.

17. The result obtained in the last example deserves attention, because it is opposed to a theory of discontinuous functions, which has received the sanction of an eminent geometer, Dr. Hermann Hankel, whose recent death at an early age is a great loss to mathematical science. In an interesting memoir ("Untersuchungen ueber die unendlich oft oscillirenden und unstetigen Functionen," Tübingen, 1870), Dr. Hankel has laid down the distinction, here adopted from him, between a system of points which completely fill a segment, and a system of points which do not completely fill any segment, but lie in loose order. [The term employed by Dr. Hankel is "zerstreut"; the use of the equivalent English words "dispersed" or "scattered" has been avoided in the present note, because they might seem to exclude the sort of condensation in the vicinity of a finite or infinite number of points, which, as we have seen in the examples (i.), (ii.), (iii.), may present itself in the case of systems of points in loose order.] Dr. Hankel then asserts (see p. 26) that, when a system of points is in loose order on a line, the line may be so divided as to make the sum of the segments containing the points less than any assignable line. The proof of this assertion is, in effect, as follows:—Divide the line into segments, of which each contains a point of the system, and imagine each segment to be diminished to its n^{th} part, yet so as still to have upon it the point of the system which it contained before. The sum of the segments can thus be made less than the n^{th} part of the whole line; *i.e.*, less than any line that can be assigned, because we may suppose n as great as we please. It must be conceded that this demonstration is rigorous, if the number of points in the system is finite; but the construction indicated ceases to convey any clear image to the mind, as soon as the number of points becomes infinite. If we are allowed to divide the line from 0 to 1, in example (iii.), in such a manner as to include every point (P_{s+1}) in a seg-

ment of its own, these segments, in the vicinity of the points P_s, will have to be less than any line that can be assigned; and, if such a mode of division is admissible, it is difficult to see why it should not also be considered admissible so to divide the line as to include every rational point in a segment of its own: in which case Dr. Hankel's proposition would extend to systems of points in close order, as well as to systems in loose order. But whether we do or do not admit the truth of Dr. Hankel's proposition, the use which he makes of it (p. 31) to establish the applicability of Riemann's criterion to a certain class of functions would seem to be erroneous. To prove that Riemann's condition of integrability is satisfied for a given discontinuous function, we have to show that, given any finite quantity d, however small, the sum of the segments, which, *in any division whatever of norm* d, contain the points of discontinuity, is evanescent with d. And it is evident that this cannot be shown, if we confine ourselves to considering modes of division in which some of the segments are from the very beginning assumed to be less than any quantity that can be assigned.

While, therefore, we may safely admit the theorem that no function can be integrable which has discontinuities, surpassing a given quantity σ, at an infinite number of points forming a system in close order; the converse assertion that, when the system of points of discontinuity is in loose order, the function is integrable, would seem to be established by no satisfactory demonstration, and to be negatived by the result obtained in example (v.)

18. Another proposition, contained in the same memoir (p. 28), appears open to a similar objection. It may be admitted that a function $f(x)$ having discontinuities, which surpass a given quantity σ however small, only at points which form a system in loose order, is necessarily continuous over finite portions of any interval however small. But it would seem to be untrue that such a function is necessarily continuous in the vicinity of any one of its points of discontinuity. If, for example, $f\left(\frac{1}{a_1}+\frac{1}{a_2}\right)=1$, and $f(x)=0$, for every other value of x, it is evident that, however small the given quantity ϵ may be, the difference $f\left(\frac{1}{a_1}+\epsilon\right)-f\left(\frac{1}{a_1}+\delta\right)$ oscillates an infinite number of times between the values 0 and 1, as δ decreases from ϵ to 0; *i.e.*, the function $f(x)$ is discontinuous in the vicinity of the point $\frac{1}{a_1}$ to the right.

19. We add a few remarks which may serve still further to illustrate the meaning and use of Riemann's theorem.

(i.) The problem, "Given a system of points upon an interval (a, b),

to find, among all divisions of norm d, that in which the segments containing the points have the maximum sum," is perfectly determinate. We may say that a point of the system is *isolated*, when it is separated from the next preceding and next following point by a distance $> 2d$. Similarly a group of points may be said to be isolated, when the distance between any two consecutive points of the group is less than $2d$, but the distance between the extreme points of the group, and those which immediately precede and follow it, is greater than $2d$. It is evident that, for any given value of d, the given system of points resolves itself into a finite number of isolated groups. The first and last point of each group determine a segment; on either side of each of these segments, and on either side of each isolated point, we may place a segment equal to d. The sum of the segments thus obtained is the maximum sum required.

It will be observed that in this solution each point of the system is regarded as double; *i.e.*, as capable of affecting two segments at once, one on each side of it. If the discontinuity of a function at any point can be removed by changing the value of the function at that point only, for example, if $f(x-0) = a$, $f(x) = a+\sigma$, $f(x+0) = a$, the point must be regarded as single (its contribution to the difference Θ of Art. 2 would be only $\sigma \times d$). But if the values of the function preceding and following the point of discontinuity are different (*i.e.*, if $f(x-0) = a$, $f(x+0) = a+\sigma$), the point of discontinuity produces a double effect, its contribution to the difference Θ being $2\sigma \times d$. Similarly, in the case of functions which, like $\cos\left(\frac{\pi}{x}\right)$ in the vicinity of the origin, admit of an infinite number of maxima and minima within a finite interval, the contribution to Θ of each point at which there is a maximum, or minimum, is two-fold. For the practical application of Riemann's criterion, the distinction between points producing a one-fold effect and those producing a two-fold effect is immaterial.

20. (ii.) When a function, which is discontinuous but never infinite, does not admit of integration between the limits a and b, the symbol $\int_a^b f(x)\,dx$ becomes indeterminate. But the maximum and minimum values attributable to that symbol are perfectly determinate; and if it should become advisable to attribute a definite value to the symbol, we might select for that purpose the arithmetical mean between these two extreme values. If, for continually decreasing values of d, we calculate the corresponding maximum values of the sum S of Art. 2, these values will, as shall now be shown, converge to a determinate limit A. And similarly the successive minimum values of S will converge to a determinate limit B, different from A in the case under consideration.

The difference A−B is, of course, the limit of the successive differences Θ.

From the two sets of inequalities

$$A(d_1) > A(d_2) > A(d_3) > \ldots\ldots,$$
$$B(d_1) < B(d_2) < B(d_3) < \ldots\ldots,$$

combined with the inequality $A(d_n) > B(d_n)$, which holds for any value of n however great, we infer that each of the two series,

$$A(d_1)-A(d_2),\quad A(d_2)-A(d_3),\quad A(d_3)-A(d_4),\ \ldots\ldots,$$
$$B(d_2)-B(d_1),\quad B(d_3)-B(d_2),\quad B(d_4)-B(d_3),\ \ldots\ldots,$$

consists of positive terms, and that, however many terms of either series we add together, we can never surpass $A(d_1)-A(d_\infty)$ in the first, and $B(d_\infty)-B(d_1)$ in the second; *i.e.*, in neither of them can we ever surpass $A(d_1)-B(d_1)$. But if a series of positive terms be such that the sum of any number of its terms, however great, can never surpass a given finite quantity, the sum of the first n terms of the series converges to a finite and determinate limit, when n is increased without limit (see Riemann, Vorlesungen, pp. 39, 40). The sums $A(d_1)-A(d_n)$, $B(d_n)-B(d_1)$, therefore converge to finite and determinate limits; or, which is the same thing, the two series of terms

$$A(d_1),\quad A(d_2),\quad A(d_3),\ \ldots\ldots,$$
$$B(d_1),\quad B(d_2),\quad B(d_3),\ \ldots\ldots,$$

converge to finite and determinate limits.

If, for example, the function $f(x)$ have the value σ, at every point of the system considered in Art. 16, example (v.), and the value $\sigma_2 < \sigma_1$ at every other point; we shall find

$$B = \sigma_2,\quad A = \sigma_2 + (\sigma_1 - \sigma_2) \times E\left(\frac{1}{m}\right).$$

21. (iii.) Riemann's criterion of integrability is applicable to the case of any multiple integral extended over a finite space. For example, in the case of a triple integral, we must imagine the whole space of the integration divided into small spaces such that any one of them could be comprehended within a sphere of a diameter d; and any such division into spaces is a division of norm d. The criterion of integrability, then, is that, in any division whatever of norm d, the sum of the spaces in which the ordinate-differences surpass a given quantity σ, must diminish without limit with d. The ordinate-difference of any space is, of course, the difference between the greatest and least values of the function within the space.

Considering, for simplicity, the case of two dimensions only, we observe that the space of integration may not only contain points of discontinuity finite or infinite in number, but may be intersected by curves

of discontinuity. The function may have values differing by a finite quantity on either side of such a curve; or its values at points along the curve may be discontinuous, or both of these kinds of discontinuity may be combined at the same curve. If $L(\sigma)$, the total length of the curves at which the discontinuities surpass σ, be finite, the function can be integrated over the given space; since, if we draw curves parallel to the curves of discontinuity and at a distance d from them on either side, the area of the channel-like spaces thus obtained will be $2dL(\sigma)$, and will surpass the greatest sum of spaces, including the curves in any division of norm d. But the function may be integrable even if the total length of the curves of discontinuity is infinite; because an infinite number of contiguous curves may be enclosed in one and the same channel. And, provided that the curves can all be included in channels of which the length is L, and of which the breadth δ is comminuent with d, the condition that $L \times \delta$ should be comminuent with d, will suffice to ensure the integrability of the function.*

On the Higher Singularities of Plane Curves.

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The ordinary singularities of a plane curve are its double points and double tangents, its stationary points and stationary tangents; or, as they have been also called, its nodes and links, its cusps and inflexions. The fundamental theorem, that any of the so-called higher singularities of a plane curve may be regarded as equivalent to a certain number of ordinary singularities of each of these four kinds, has been enunciated by Professor Cayley, who has also given a method for determining in every case the four indices δ, τ, κ, ι, proper to any given singularity.

Several enquiries, which appear to possess some interest, are suggested by this theorem. Among them we may mention the two following—

(1). It is important to prove that the indices of singularity, as defined by Professor Cayley, satisfy the equations of Plücker; and that the "genus" or "deficiency" of the plane curve is correctly given by these indices.

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* This Paper, though it was not read, was offered to the Society and accepted in the usual manner.

(2). It is also of interest to examine whether any given singularity can be actually formed by the coalescence of the ordinary singularities to which it is regarded as equivalent; in other words, whether a singularity of which the indices are δ, τ, κ, ι, and which is therefore to be regarded as equivalent to δ double points, τ double tangents, κ cusps, and ι inflexions, possesses a penultimate form, in which all these singularities exist, distinct from one another, but infinitely close together.

The present paper relates chiefly to the first of these enquiries; the second is reserved for a future communication.

1. Consider a plane curve C of order m and class n, defined by an equation $F(p, q) = 0$ between the parameters of two pencils, of which the corresponding rays intersect on C, and which are represented by equations of the form $p\,(QP) + (QR) = 0$, $q\,(PQ) + (PR) = 0$; P, Q, R denoting the three vertices of a triangle, $(PQ) = 0$, $(QR) = 0$, $(PR) = 0$ the equations of its sides. It is convenient to suppose that Q and P, the centres of the two pencils, have no speciality of position with regard to C; or, more precisely, that neither Q nor P lies on the curve, nor on any singular line appertaining to the curve. Under the general name of singular lines we include (1) lines joining two singular points, (2) singular tangents, (3) tangents at singular points, (4) tangents passing through a singular point; we shall also suppose that PQ is not a tangent to C, and does not pass through any singular point. Thus to every finite value of p there will correspond m finite values of q, and *vice versâ;* and, in particular, to any singular point on the curve there will correspond a finite pair of values of p and q. To an infinite value of q there will correspond m infinite values of p, and *vice versâ;* these answer to the m intersections of PQ with the curve, no two of which, by hypothesis, are coincident. We may, if we please, project the line PQ to an infinite distance, and regard p and q as Cartesian coordinates; we prefer, however, for our present purpose, to consider them as parametric ratios; *i. e.*, as purely numerical quantities (real or complex).

2. Let $f(q)$ be the discriminant of $F(p, q) = 0$, considered as an equation of the order m in p; we may suppose the coefficient of p^m, which is certainly different from zero, to be unity. The first polar of P with regard to C is $\frac{dF}{dp} = 0$, and $f(q)$ is the resultant of the elimination of p from F and $\frac{dF}{dp}$, so that the roots of $f(q) = 0$ are the parameters of the lines drawn from P to the points of intersection of C with the first polar of P. Attending to the suppositions which have been made as to the situation of P and Q relatively to the curve C, we infer (α) that $f(q)$ has no infinite roots, and is therefore of the

full order $m(m-1)$ in q; (β) that $f(q)$ has n, and only n, non-multiple roots q'; (γ) that for each of these n roots q' the equation $F(p, q') = 0$ acquires two equal roots p', its remaining roots being all different from p', and from one another; (δ) that q' is not a multiple root of the equation $F(p', q) = 0$. The n sets (p', q') give the n points of contact of tangents from P; the remaining factor of $f(q)$, viz., $f_1(q) = f(q) \div \Pi(q-q')$, consists exclusively of multiple factors, and appertains to the singular points of the curve. The index of its order, *i. e.*, $m(m-1)-n$, we may term the *total discriminantal index of the singular points of the curve.* Let q_0 be a root of $f_1(q) = 0$ of multiplicity ν; the equation $F(p, q_0) = 0$ has but one multiple root; let this be p_0, and let its multiplicity be μ; then (p_0, q_0) is a singular point O on the curve, of which the *order* (*i. e.*, the least number of points in which it is cut by any straight line passing through it) is μ, and of which ν may be termed the *discriminantal index.* It is evident that the number of singular points is equal to the number of unequal roots of $f_1(q) = 0$, and that the total discriminantal index is equal to the sum of the discriminantal indices of the separate singular points. We shall presently (Art. 8) see that the discriminantal index of a singular point can in general be further subdivided into parts, appertaining respectively to the different branches of the curve which pass through the point, taken singly, and in pairs.

3. It is a well-known theorem of Cauchy, that so long as the analytical modulus of $q-q_0$ is less than the least of the modules of any of the quantities q_1-q_0, where q_1 is any root of $f(q) = 0$ other than q_0, the m roots of the equation $F(p, q) = 0$ are developable in convergent series of the form

$$(A) \ldots\ldots p-p_0 = A+A_0(q-q_0)+A_1(q-q_0)^{\alpha_1}+A_2(q-q_0)^{\alpha_2}+\ldots\ldots,$$

the exponents $\alpha_1, \alpha_2, \ldots\ldots$ being rational and positive numbers, which satisfy the inequalities $1 < \alpha_1 < \alpha_2 < \ldots\ldots$ Of the equations (A), $m-\mu$ give the values of p corresponding to the $m-\mu$ points not in the vicinity of O, in which C is cut by the line (q). The series in the right hand members of these $m-\mu$ equations we shall designate by $\bar{A}_1, \bar{A}_2, \ldots \bar{A}_{m-\mu}$: we observe that in them the quantities A are all different from one another and from zero; because (q_0), not being a singular line, intersects C in $m-\mu$ points, which are different from one another and from O; also, in these equations, the exponents $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots$ are integral. In the remaining μ equations, which give the developments appertaining to the branches of C that pass through O, the quantities A are all equal to zero: these equations divide themselves into groups of conjugate equations, the equations of any one group being of the

type $$p-p_0 = B_0(q-q_0)+B_1\omega^{\beta_1}(q-q_0)^{\frac{\beta_1}{\Delta}}+\ldots\ldots,$$

where the numerators β are positive, integral, and increasing; Δ is less than β_1, and is the least common denominator of the fractional exponents; ω is any root of $\omega^\Delta = 1$: so that, if we use one and the same value of the radical in all the Δ equations of the group, they will differ from one another only by containing different values of ω; each of the μ equations defines a branch of the curve passing through O. If $\Delta = 1$, the branch is linear or of order 1; if $\Delta > 1$, the Δ conjugate equations are regarded by Professor Cayley as defining Δ partial branches forming a single superlinear branch of order Δ; in every case the sum of the orders of the branches is equal to the order of the point, *i.e.*, $\Sigma\Delta = \mu$. The coefficients B_0 are all different from zero, and the indices $\frac{\beta_1}{\Delta}$ are all greater than unity, because neither (p_0) nor (q_0) is one of the tangents at O; but these coefficients and indices are not necessarily different in two developments belonging to two different linear or superlinear branches—indeed any two such developments may coincide for any finite number of terms; and to ascertain the true nature of any singular point it is indispensable to continue the developments until they all become different from one another. The series in the right-hand members of the μ equations we denote by $\overline{B}_1, \overline{B}_2, \overline{B}_3 \ldots\ldots \overline{B}_\mu$.

4. The series $\overline{A}$ and $\overline{B}$ of the preceding article are *absolutely* convergent within the assigned limits; *i.e.*, any one of these series would continue to be convergent within these limits if its terms were replaced by their analytical modules. For the multiplication of two absolutely convergent series we have the theorem:—

"If the product of two given absolutely convergent series, proceeding by ascending powers of a variable, be arranged in a series proceeding in the same manner, this series is absolutely convergent for all values of the variable for which the given series are absolutely convergent, and its sum is equal to the product of the sums of the given series." (Cauchy, "Analyse Algébrique," cap. vi.)

Multiplying together the m series $p-p_0-\overline{A}$, and $p-p_0-\overline{B}$, we obtain, by virtue of this theorem, the equation

$$F(p, q) = \Pi(p-p_0-\overline{A}) \times \Pi(p-p_0-\overline{B}).$$

This equation is an identity; *i.e.*, if the multiplication be actually effected on the right-hand side, all powers of $q-q_0$ above the m^{th} will disappear, and the terms that remain will be precisely the terms of $F(p_0+\overline{p-p_0}, q_0+\overline{q-q_0})$ or $F(p, q)$. But an arithmetical equality between the two sides of the equation subsists only so long as the analytical modulus of $q-q_0$ does not surpass the limit assigned in Cauchy's theorem (Art. 3). Subject to the same limitation, $f(q)$ is equal to the product of the squares of the differences of every two of the series $\overline{A}$ and $\overline{B}$.

5. The number of the intersections at any point O of two branches of the same curve, or of different curves, which pass through the point, and which are there represented by equations of the form

$$p^{(1)}-p_0 = B_0^{(1)}(q-q_0)+ \ldots\ldots$$
$$p^{(2)}-p_0 = B_0^{(2)}(q-q_0)+ \ldots\ldots$$

is defined by Professor Cayley to be the number which expresses the order of evanescence of $p^{(1)}-p^{(2)}$, *i.e.*, the integral or fractional exponent λ for which $\frac{p^{(1)}-p^{(2)}}{(q-q_0)^\lambda}\ \lambda$ has a finite limit, when $q-q_0$ is diminished without limit. We may justify this definition by proving that, whenever two curves C_1 and C_2 have a multiple intersection at any point, its multiplicity is correctly obtained by adding together the numbers (as thus defined) of the intersections of each branch of C_1 by each branch of C_2. If we suppose (as we may do) that the points P and Q have no speciality of position with regard to the curves C_1 and C_2 considered as one curve, the resultant $\Phi(q)$ of the equations $C_1(p, q) = 0$ and $C_2(p, q) = 0$ is of the order $m_1 \times m_2$; and if μ_1 branches of C_1 and μ_2 branches of C_2 pass through O, we shall have, for C_1, $m_1-\mu_1$ equations $\overline{A}^{(1)}$, and μ_1 equations $\overline{B}^{(1)}$; and similarly, for C_2, $m_2-\mu_2$ equations $\overline{A}^{(2)}$, and μ_2 equations $\overline{B}^{(2)}$. Denoting by $\Pi.(\overline{B}^{(1)}-\overline{B}^{(2)})$ the product of the $\mu_1\times\mu_2$ differences obtained by subtracting them in succession, each series $\overline{B}^{(2)}$ from each series $\overline{B}^{(1)}$, and by λ the number of intersections, as above defined, of any one branch of C_1 by any one branch of C_2, we see that the limit of $\Pi(\overline{B}^{(1)}-\overline{B}^{(2)}) \div (q-q_0)^{\Sigma\lambda}$ is finite. But $\Phi(q) = \Pi(\overline{B}^{(1)}-\overline{B}^{(2)})$, the sign of multiplication now extending to all the $m_1 \times m_2$ differences obtained by considering the m_1 series $\overline{A}^{(1)}$ and $\overline{B}^{(1)}$, and the m_2 series $\overline{A}_2$ and $\overline{B}_2$; and of these $m_1 \times m_2$ differences, none, except the $\mu_1 \times \mu_2$ differences already considered, are evanescent with $q-q_0$ (for the hypothesis that P and Q have no speciality of position with regard to the system of the two curves C_1 and C_2 implies that none of the constants $A^{(1)}$ can be equal to any of the constants $A^{(2)}$). Hence $\Sigma\lambda$ is the multiplicity of the factor $q-q_0$ in $\Phi(q)$; *i.e.*, since (p_0, q_0) is the only intersection of C_1 and C_2 which lies on (q_0), $\Sigma\lambda$ is the multiplicity of that intersection.

If we regard the equation $F(p, q) = 0$ as determining a correspondence of points on a line, the coincidences of corresponding points (except indeed the coincidences $p = q = \infty$) answer in number and multiplicity to the intersections of C by the straight line $p = q$. We are thus led to a theorem given by M. Zeuthen (Bulletin des Sciences Mathématiques, Vol. V., p. 186).

6. As it is only the hypothesis that the points P and Q have no

speciality of position with regard to C, which gives us a right to assert that every one of the developments $\overline{B}$ contains a term linear in $q-q_0$, and no term in which the exponent of $q-q_0$ is less than unity, it is worth while to see how far the results of the preceding article can be depended on when this hypothesis is dispensed with. It will be found that if (p_0) is one of the tangents at O, *i.e.*, if one, or more, of the coefficients B_0 is zero, the discriminantal index of the point is still equal to the order of evanescence of $\Pi(\overline{B}_i-\overline{B}_j)^2$. But this conclusion would no longer hold, if (q_0) were one of the tangents at O. In this case the developments appertaining to the branches to which (q_0) is a tangent would contain powers of $q-q_0$ inferior to unity; and the order of evanescence of $\Pi(\overline{B}_i-\overline{B}_j)^2$ would exceed the discriminantal index of O by τ, if $n-\tau$ is the number of tangents other than (q_0) which can be drawn to the curve from P. But the multiplicity of the intersection of two different curves, at a point which is singular for one or both of them is correctly obtained by the process of Art. 5, even when the developments contain positive powers of $(q-q_0)$ inferior to unity. Thus, in the curve $p-p_0=(q-q_0)^{\frac{1}{a}}$, a being an integer, the order of evanescence of $\Pi[\overline{B}_i-\overline{B}_j]^2$ is $a-1$, whereas the point is not a singular point at all, and has consequently a discriminantal index equal to zero: its tangent (q_0) however is a singular tangent, and counts as $a-1$ tangents drawn from P. On the other hand, if we consider the two curves $(p-p_0)=(q-q_0)^{\frac{1}{a}}$, $p-p_0=(q-q_0)^{\frac{1}{b}}$, in which a and b are both integers and $b<a$, the order of evanescence of $\Pi(\overline{B}_i-\overline{B}_j)^2$ is b; and this is the multiplicity of the intersection at (p_0, q_0).

7. We can now prove that the discriminantal index of the singular point O is equal to twice the number of the intersections of C by itself at that point; and, again, that this discriminantal index is equal to the number of the intersections at the same point of C by its first polar with regard to any point not having a special position. For (1), considering the μ equations $\overline{B}$, we see that twice the number of intersections of C by itself at the point (p_0, q_0), is the order of evanescence of $\Pi(\overline{B}_i-\overline{B}_j)^2$, the sign of multiplication extending to all the $\frac{1}{2}\mu(\mu-1)$ differences; or, observing that $f(q)\div\Pi(\overline{B}_i-\overline{B}_j)^2$ is a product of $\frac{1}{2}m(m-1)-\frac{1}{2}\mu(\mu-1)$ squared differences, none of which vanish with $q-q_0$, twice the number of intersections of C by itself at the point (p_0, q_0) is equal to the order of evanescence of $f(q)$ with $q-q_0$, *i.e.*, to the discriminantal index ν. And (2), since the polar of P is $\frac{dF}{dp}=0$, and since the resultant of $F=0$ and $\frac{dF}{dp}=0$ is $f(q)$, we infer (Art. 5)

that ν is the number of intersections at (p_0, q_0) of C by its polar with regard to P.

8. Considering a superlinear branch, of which the component branches are defined by the Δ equations

$$(\Delta) \ldots\ldots p - p_0 = B_0 (q - q_0) + B_1 \omega^{\beta_1} (q - q_0)^{\frac{\beta_1}{\Delta}} + \ldots,$$

let Δ_1 be the greatest common divisor of Δ and β_1; and if γ_1 is the first of the numbers β which is not divisible by Δ_1, let Δ_2 be the greatest common divisor of Δ_1 and γ_1; if, again, γ_2 is the first of the numbers β which is not divisible by Δ_2, let Δ_3 be the greatest common divisor of Δ_2 and γ_2, and so on continually. Since the numbers β have no common divisor with Δ, we shall at last arrive in the series $\Delta_1, \Delta_2, \Delta_3 \ldots\ldots$ at a term equal to unity, when the series will terminate: and twice the number of the intersections of the superlinear branch by itself will be expressed by the formula

$$2N = \gamma (\Delta - \Delta_1) + \gamma_1 (\Delta_1 - \Delta_2) + \gamma_2 (\Delta_2 - \Delta_3) + \ldots\ldots,$$

in which γ is written for β_1. For if ω denote any given root of the equation $\omega^\Delta = 1$, of its remaining roots x there are $\Delta_i - 1$ which verify the equations $x^\gamma = \omega^\gamma$, $x^{\gamma_1} = \omega^{\gamma_1}$, $\ldots\ldots$ $x^{\gamma_{i-1}} = \omega^{\gamma_{i-1}}$; because Δ_i is the greatest common divisor of $\Delta, \gamma, \gamma_1 \ldots \gamma_{i-1}$: similarly there are $\Delta_{i+1} - 1$ roots other that ω_1 which verify the same equations, and in addition the equation $x^{\gamma_i} = \omega^{\gamma_i}$. Thus, of the $\Delta(\Delta - 1)$ differences obtained by subtracting each of the series (Δ) in turn from every other, there are $\Delta(\Delta_i - \Delta_{i+1})$ which are of the order $\frac{\gamma_i}{\Delta}$; *i.e.*, $2N = \Sigma\gamma_i (\Delta_i - \Delta_{i+1})$. The value of N depends, therefore, not on every exponent in the series (Δ), but only on certain *critical* exponents $\frac{\gamma}{\Delta}$, in the denominators of which, when reduced to their lowest terms, a new factor appears for the first time. The number 2N, which is the "discriminantal index" of the superlinear branch is, not itself necessarily even, but the difference $2N - (\Delta - 1)$ is always even, since we have

$$2N - (\Delta - 1) = (\gamma - 1)(\Delta - \Delta_1) + (\gamma_1 - 1)(\Delta_1 - \Delta_2) + \ldots,$$

and in this expression, if Δ is uneven, so also are $\Delta_1, \Delta_2 \ldots\ldots$; if Δ is even, let Δ_i be the first of the numbers $\Delta_1, \Delta_2 \ldots$ which is uneven; then γ_{i-1} is uneven, and so are all the subsequent numbers $\Delta_{i+1}, \Delta_{i+2}, \ldots\ldots$. In either case, therefore, every term in the expression of $2N - (\Delta - 1)$ is even.

Again, if two superlinear branches of the orders Δ and Δ' have the same tangent, let $(q - q_0)^h$ be the lowest power of $q - q_0$ which has not the same coefficient B in the two sets of series (Δ) and (Δ'): it may, of course, in one of these sets have a zero coefficient. Then the terms of lower exponent are common to the two sets; and if the exponents be

reduced to their least common denominator, these initial terms will be of the form

$$B_0(q-q_0)+B'\theta^{\alpha_1}(q-q)^{\frac{\alpha_1}{d}}+B\theta^{\alpha_2}(q-q)^{\frac{\alpha_2}{d}}\ldots\ldots+B_i\theta^{\alpha_i}(q-q_0)^{\frac{\alpha_i}{d}},$$

where d is a common divisor of Δ and Δ', θ is any root of $\theta^d=1$, and $\frac{\alpha_i}{d}$ is the exponent next inferior to h. The number of intersections of the two superlinear branches is then

$$N'=h\frac{\Delta\Delta'}{d}+\frac{\Delta\Delta'}{d^2}[\sigma(d-d_1)+\sigma_1(d_1-d_2)+\ldots\ldots],$$

the numbers $\sigma, \sigma_1, \sigma_2, \ldots\ldots d_1, d_2, \ldots\ldots$ (of which in particular $\sigma=\alpha_1$) being determined from the series of exponents $\frac{\alpha}{d}$, in the same way that the numbers $\gamma, \gamma_1, \ldots\ldots \Delta_1, \Delta_2, \ldots\ldots$ were determined from the series of exponents $\frac{\beta}{\Delta}$. For, if θ represent a given root of the equation $\theta^d=1$, the $\frac{\Delta}{d}$ roots of the equation $\omega^\Delta=1$, which satisfy the equation $\omega^{\frac{\Delta}{d}}=\theta$, will give the same initial terms; and we may thus divide the equations (Δ) into $\frac{\Delta}{d}$ groups, each containing d equations; the equations of the same group differing from one another by containing different values of θ, but the different groups not differing from one another, so far as the initial terms are concerned. Similarly we may divide the equations (Δ') into $\frac{\Delta'}{d}$ groups. Considering only one group of each set, we find (by the same reasoning as before) for the order of the product of the $d\times d$ differences obtained from them, the expression

$$hd+\sigma(d-d_1)+\sigma_1(d_1-d_2)+\ldots\ldots,$$

the additional term hd appearing because we have now to take into account the d differences in which all the initial terms vanish: the result, multiplied by $\frac{\Delta}{d}\times\frac{\Delta'}{d}$, gives the value of N'.

Lastly, when two superlinear branches have not the same tangent, the number of their intersections is evidently $N''=\Delta\Delta'$. By means of these formulæ the discriminantal indices of the branches at any singular point, taken by themselves or in pairs, may always be obtained as soon as the developments appertaining to the branches have been found. The sum of these separate discriminantal indices is of course the discriminantal index of the point, or $\nu=2\Sigma N+2\Sigma N'+2\Sigma N''$.

9. Every singular point of a plane curve is regarded by Professor Cayley as being equivalent in a certain manner to δ common nodes and

κ common cusps; and, correlatively, every singular tangent as equivalent to τ double tangents and ι inflexional tangents. For any superlinear branch of order Δ passing through a singular point, the *cuspidal index* κ is by definition $\Delta-1$; thus, for a linear branch $\kappa = 0$. The cuspidal index of a singular point is the sum of the cuspidal indices of the several superlinear branches passing through it; so that, for any singular point, $\kappa = \Sigma(\Delta-1) = \mu-\lambda$, if μ is the order (Art. 2) of the point, and λ the number of distinct linear, or superlinear, branches passing through it. The *nodal index* δ for a singular point, and for its branches, taken singly or in pairs, is defined, not directly, but by equating $2\delta+3\kappa$ to the discriminantal index; thus, for any superlinear branch of order Δ, we have

$$2\delta = (\gamma-3)(\Delta-\Delta_1)+(\gamma_1-3)(\Delta_1-\Delta_2)+\ldots\ldots,$$

which is always even (Art. 8), and positive, except when $\gamma = 3$, $\Delta = 2$, in which case $\delta = 0$, and the superlinear branch is a common cusp.

For τ and ι we have correlative definitions.

10. Adopting these definitions, we have now to prove that the numbers $\Sigma\delta$, $\Sigma\kappa$, $\Sigma\tau$, $\Sigma\iota$ (the summations extending to all the singularities of the curve) satisfy the equations of Plücker, and further that the deficiency of the curve is correctly given by the formula

$$\mathrm{H} = \tfrac{1}{2}(m-1)(m-2)-\Sigma\delta-\Sigma\kappa.$$

It is sufficient to establish the four equations,

$$\text{(i.)}\ \ldots\ldots n = m(m-1)-2\Sigma\delta-3\Sigma\kappa,$$
$$\text{(ii.)}\ \ldots\ldots m = n(n-1)-2\Sigma\tau-3\Sigma\iota,$$
$$\text{(iii.)}\ \ldots\ldots \mathrm{H} = \tfrac{1}{2}(m-1)(m-2)-\Sigma\delta-\Sigma\kappa,$$
$$\text{(iv.)}\ \ldots\ldots \mathrm{H} = \tfrac{1}{2}(n-1)(n-2)-\Sigma\tau-\Sigma\iota,$$

because the three equations (i.), (ii.), and (iii.) = (iv.) are equivalent to the six equations of Plücker. But the equation (i.) has been already proved; for we have found (Art. 2) that $n = m(m-1)-\Sigma\nu$; and by definition $\Sigma\nu = \Sigma(2\delta+3\kappa)$. The equation (ii.) is the correlative of (i.) and needs no separate proof. In the equations (iii.) and (iv.) it is important to take a definition of H which does not involve any special supposition as to the nature of the singularities appertaining to the curve. The simplest, though not the most direct, course is to adopt the method of Riemann, and to define $2\mathrm{H}+1$ as the index of multiplicity of connexion of the m-leaved spirally connected surface [Q], which is such that if the complex values of q be represented upon it in the usual manner, p may be regarded as a one-valued function of q. In any such surface the index of multiplicity of connexion $2\mathrm{H}+1$, the number of leaves m, and the number of *spires* (spiral points, *windungs-punkte*) N, are connected by the equation $\mathrm{N} = 2\mathrm{H}+2m-2$. This equation Riemann

himself demonstrates by comparing the values of certain contour-integrals (*Theorie der Abelschen Functionen*, Art. 7). But he observes that it is entirely independent of considerations of magnitude, and that it belongs properly to the geometry of situation. The demonstration of it from this point of view, which has been given by M. Neumann (*Vorlesungen* p. 309, § 99), is also independent of any supposition as to the special nature of the singularities of the curve C; and is therefore available for our present purpose. But we may observe that the algebraical demonstration of the same equation, which is given by MM. Clebsch and Gordan (in their *Theorie der Abelschen Functionen*, p. 54, § 16), would here be inadmissible, because in that demonstration it is expressly supposed that the singular points of C are only common nodes and cusps. (See the note at p. 11, *loc. sit.*)

It is not difficult to find the number of spires N on the surface [Q]. There is a one-fold spire for every tangent from P to C; for, if (p_0, q_0) be the point of contact of any such tangent, we have for values of q in the vicinity of q_0 two conjugate developments of the type

$$(p-p_0) = B_1(q-q_0)^{\frac{1}{2}} + B_2(q-q_0)^{\frac{2}{2}} + \ldots\ldots,$$

in which B_1 is different from zero; all the other developments (Art. 3) being of the type (A), because the point P has no speciality of position. Again, there is a $(\Delta-1)$-fold spire for any singular branch which is superlinear and of order Δ; this is apparent from the form of the Δ developments appertaining to the branch (see Riemann, *loc. cit.*, Art. 6; M. Puiseux, *Liouville*, 1st series, Vol. XV. pp. 384–404).

We have therefore $N = n + \Sigma(\Delta-1) = n + \Sigma\kappa$, and Riemann's equation becomes

$$n + \Sigma\kappa = 2H + 2(m-1);$$

or, since

$$n + \Sigma\kappa = m(m-1) - 2\Sigma\delta - 2\Sigma\kappa,$$

$$H = \tfrac{1}{2}(m-1)(m-2) - \Sigma\delta - \Sigma\kappa,$$

which is the equation (iii.) Again, it is an immediate consequence of Riemann's definition of the number H (see his *Abelsche Functionen*, Art. 11) that this number remains unchanged by any unicursal transformation of the equation $F(p, q) = 0$. But (as has been already observed by MM. Clebsch and Gordan) any tangential equation of the curve C may be regarded as an unicursal transformation of the equation $F(p, q) = 0$, because the points and tangents of a curve correspond to one another one to one. The equation (iii.), therefore, involves the equation (iv.); a result which, as we have seen, implies that the six equations of Plücker are satisfied by the numbers $\Sigma\delta$, $\Sigma\kappa$, $\Sigma\tau$, $\Sigma\iota$.

11. The indices τ and ι appertaining to any superlinear branch at a singular point, and the number of tangents common to two osculating superlinear branches, may be ascertained directly from the point-equations $\overline{B}$, without actually forming the corresponding line-equations.

To prove this, we shall establish a relation which subsists between certain terms in the two sets of equations.

If Q and R are given constants, $p = Qq + R$ is the equation of a straight line in the system of parametric point-coordinates which we have been employing. In passing to line-coordinates, we may take Q and R as the coordinates of this straight line; and we may regard Q and R as the parameters of two ranges of points, lying on the lines PQ, PR, respectively, and represented by equations of the form

$$Q(P) + (Q) = 0, \quad R(P) + (R) = 0;$$

the line $p = Q_0 q + R_0$ or (Q_0, R_0) being the line joining the points determined in the two ranges by the values Q_0, R_0 of the parameters. If to the hypotheses of Art. 1 we add the supposition that PR is not a tangent to C, and does not pass through any singular point of C, the line-equation of C, which we may represent by $\Phi(Q, R) = 0$, will have the same sort of freedom from speciality which has been already attributed to the point-equation $F(p, q) = 0$. The parameters of the tangent to C at the point (p, q) are

$$Q = -\left(\frac{dF}{dq}\right) \div \left(\frac{dF}{dp}\right), \quad R = p - qQ.$$

Let (p, q) be a point lying on the branch $\overline{B}$, of which the point-equation is $p - p_0 = B_0(q - q_0) + B_1 \omega^{\beta_1}(q - q_0)^{\frac{\beta_1}{\Delta}} + \ldots\ldots;$

and suppose (p, q) different from (p_0, q_0), but sufficiently near to it (Art. 3) to ensure the convergence of the m series A and $\overline{B}$. Writing

$$F(p, q) = M \times (p - p_0 - \overline{B}),$$

where M is a product of factors, none of which can vanish at the point (p, q), because no singular point other than (p_0, q_0) exists within the range of values attributed to q, we find

$$Q = \left(\frac{d\overline{B}}{dq}\right), \quad R = p - q\left(\frac{d\overline{B}}{dq}\right).$$

Putting $$B_0 = Q_0, \quad p_0 - q_0 B_0 = R_0,$$

so that (Q_0, R_0) is the tangent at (p_0, q_0) to $\overline{B}$, we obtain the equations

$$Q - Q_0 = \frac{\beta_1}{\Delta} B_1 \omega^{\beta_1}(q - q_0)^{\frac{\beta_1}{\Delta} - 1} + \frac{\beta_2}{\Delta} B_2 \omega^{\beta_2}(q - q_0)^{\frac{\beta_2}{\Delta} - 1} + \ldots\ldots,$$

$$R - R_0 = -q_0(Q - Q_0) + \left(1 - \frac{\beta_1}{\Delta}\right) B_1 \omega^{\beta_1}(q - q_0)^{\frac{\beta_1}{\Delta}} + \left(1 - \frac{\beta_2}{\Delta}\right) B_2 \omega^{\beta_2}(q - q_0)^{\frac{\beta_2}{\Delta}} + \ldots\ldots,$$

which determine the parameters Q, R of the tangent at any point of $\overline{\mathrm{B}}$. If we further write

$$\omega\,(q-q_0)^{\frac{1}{\Delta}}=\xi,\quad \mathrm{Q}-\mathrm{Q}_0=\frac{\beta_1\mathrm{B}_1}{\Delta}\times\mathrm{Y}^{\beta_1-\Delta},\quad \mathrm{R}-\mathrm{R}_0+q_0(\mathrm{Q}-\mathrm{Q}_0)=\mathrm{Z},$$

these equations become

$$\mathrm{Y}^{\beta_1-\Delta}=\xi^{\beta_1-\Delta}\left[1+\sigma_2\,\xi^{\beta_2-\beta_1}+\sigma_3\,\xi^{\beta_3-\beta_1}+\ldots\ldots\right]\quad\ldots\ldots\ldots\ldots\ (\alpha),$$

$$\mathrm{Z}=\rho_1\,\xi^{\beta_1}+\rho_2\,\xi^{\beta_2}+\ldots\ldots\quad\ldots\ldots\ldots\ldots\ldots\ (\beta),$$

where we have written σ_i for $\dfrac{\beta_i\,\mathrm{B}_i}{\beta_1\,\mathrm{B}_1}$, and ρ_i for $\left(1-\dfrac{\beta_i}{\Delta}\right)\mathrm{B}_i$. It will be observed that ω has disappeared from these equations, which therefore appertain equally to all the Δ branches composing the superlinear branch $\overline{\mathrm{B}}$. To obtain the tangential equation of that branch, *i.e.*, the expansion of $\mathrm{R}-\mathrm{R}_0$ in a series proceeding by powers of $\mathrm{Q}-\mathrm{Q}_0$, we have three operations to perform. First, we have to raise each side of the equation (α) to the power $\dfrac{1}{\beta_1-\Delta}$; we thus obtain an expansion of the form

$$\theta\mathrm{Y}=\xi\,(1+\mathrm{A}\xi^{a}+\mathrm{B}\xi^{b}+\ldots)\quad\ldots\ldots\ldots\ldots\ldots\ (\mathrm{Y}),$$

θ denoting any root of the equation $\theta^{\beta_1-\Delta}=1$. Secondly, we have to revert the series (Y), so as to obtain the series

$$\xi=\theta\mathrm{Y}\left\{1+\mathrm{A}'(\theta\mathrm{Y})^{a'}+\mathrm{B}'(\theta\mathrm{Y})^{b'}+\ldots\right\}\quad\ldots\ldots\ldots\ldots\ (\xi).$$

Lastly, we have to substitute, in the equation (β), for ξ its value given by the series (ξ); the final result being of the form

$$\mathrm{Z}=\mathrm{H}_1\,(\theta\mathrm{Y})^{h_1}+\mathrm{H}_2\,(\theta\mathrm{Y})^{h_2}+\ldots\quad\ldots\ldots\ldots\ldots\ldots\ (\mathrm{Z});$$

or, if

$$\mu=\left(\frac{\Delta}{\beta_1\,\mathrm{B}_1}\right)^{\frac{1}{\beta_1-\Delta}},$$

$$\mathrm{R}-\mathrm{R}_0=-q_0\,(\mathrm{Q}-\mathrm{Q}_0)+\mathrm{H}_1\,\mu^{h_1}\,\theta^{h_1}\,(\mathrm{Q}-\mathrm{Q}_0)^{\frac{h_1}{\beta_1-\Delta}}$$

$$+\mathrm{H}_2\,\mu^{h_2}\,\theta^{h_2}\,(\mathrm{Q}-\mathrm{Q}_0)^{\frac{h_2}{\beta_1-\Delta}}+\ldots\quad\ldots\ldots\ldots\ (\mathrm{H}).$$

12. Certain of the terms of H, and indeed precisely those critical terms upon which the determination of τ and ι depends, can be assigned *à priori* by the help of the following considerations.

(i.) If a, b, c, ... l, ... are positive and integral numbers, arranged in order of magnitude, of which l is such that it cannot be formed by addition of any multiples of the numbers which precede it, the coefficient of x^l in the expansion of $[\psi\,(x)]^{\sigma}$, where σ is any real exponent, and

$$\psi\,(x)=1+\mathrm{A}x^{a}+\mathrm{B}x^{b}+\mathrm{C}x^{c}+\ldots+\mathrm{L}x^{l}+\ldots,$$

is $\sigma\mathrm{L}$; and, in particular, if all the numbers preceding l are multiples of any number α, of which l is not itself a multiple, a supposition which

implies that l cannot be formed by addition of multiples of $a, b, c, \ldots$, l is the least exponent in the development of $[\psi(x)]^r$, which is not divisible by a.

(ii.) If the series $y = x\psi(x)$ be reverted so as to obtain the equation

$$x = y\psi_1(y) = y(1 + A_1 y^{a_1} + B_1 y^{b_1} + \ldots),$$

the exponents $a_1, b_1, c_1, \ldots$ are all formed by addition of multiples of $a, b, c, \ldots$. For, if this is not so, let h_1 be the least exponent in $\psi_1(y)$, which cannot be formed by adding multiples of $a, b, c, \ldots$; on substituting $x\psi(x)$ for y in $y\psi_1(y)$, a substitution which ought to have x for its result, we find that the coefficient of x^{h_1+1} is H_1; *i. e.*, $H_1 = 0$, or the exponent h_1 does not occur in $\psi_1(y)$. Again, if the exponent l in $\psi(x)$ cannot be formed by adding multiples of the exponents which precede it, the coefficient L_1 of y^l in $\psi_1(y)$ is $-L$; for, on making the same substitution as before, the coefficient of x^{l+1} is found to be $L_1 + L$; *i. e.*, $L_1 = -L$. And, in particular, if l is the lowest exponent in $\psi(x)$ which is not divisible by a, l is also the lowest exponent in $\psi_1(y)$ which is not divisible by a.

Let $\beta_i = \gamma_s$ be one of the critical exponents $\gamma, \gamma_1, \ldots$ considered in Art. 8; then all the differences $\beta_2 - \beta_1, \beta_3 - \beta_1, \ldots$ up to $\beta_{i-1} - \beta_1$, are divisible by Δ_{s-1}; but $\beta_i - \beta_1$ is not divisible by Δ_{s-1}. Therefore, by (i.), the coefficient of $\xi^{\beta_i - \beta_1}$ in the development of $\frac{\theta Y}{\xi}$ is $\frac{\sigma_i}{\beta_1 - \Delta}$; by (ii.) the coefficient of $(\theta Y)^{\beta_i - \beta_1}$ in the expansion of $\xi \div (\theta Y)$ is $-\frac{\sigma_i}{\beta_1 - \Delta}$, and $\beta_i - \beta_1$ is the least exponent in that expansion which is not divisible by Δ_{i-1}; finally, on substituting in the equation (β), we see that the term $H(\theta Y)^{\beta_i}$ in the development of Z can arise only from the terms $\rho_1 \xi^{\beta_i}$ and $\rho_i \xi^{\beta_i}$ in (β); its coefficient H is therefore $-\rho_1 \frac{\beta_1 \sigma_i}{\beta_1 - \Delta} + \rho_i = B_i$; and the coefficient of $\theta^{\beta_i}(Q - Q_0)^{\frac{\beta_i}{\beta_1 - \Delta}}$, or $\theta^{\gamma_s}(Q - Q_0)^{\frac{\gamma_s}{\gamma - \Delta}}$, in the expansion of $R - R_0$ is $\mu^{\beta_i} B_i$. Nor can any exponent preceding $\frac{\gamma_s}{\gamma - \Delta}$ have a numerator which is not divisible by Δ_{s-1}.

Observing that the greatest common divisor of $\beta_1 - \Delta$, and β_1, is the same as that of Δ and β_1, we infer from this result that the numbers $\Delta_1, \Delta_2, \Delta_3, \ldots$; $\gamma, \gamma_1, \gamma_2, \ldots$ are the same for the series H as for the series B; and since the numbers $\gamma, \gamma_1, \gamma_2, \ldots$ have no common divisor with Δ, neither have they with $\beta_1 - \Delta = \gamma - \Delta$; *i. e.*, $\gamma - \Delta$ is the least common denominator of the exponents of H. Hence we have, writing $\gamma - \Delta = \Delta_\prime$,

$$\iota = \Delta_\prime - 1, \quad \gamma = \Delta + \Delta_\prime = \iota + \kappa + 2,$$

$$2\tau + 3\iota = \gamma(\Delta_\prime - \Delta_1) + \gamma_1(\Delta_1 - \Delta_2) + \gamma_2(\Delta_2 - \Delta_3 + \ldots);$$

or, subtracting the discriminantal index $2\delta+3\kappa$,

$$(2\tau+3\iota)-(2\delta+3\kappa)=\Delta_1^2-\Delta^2,$$
$$\tau-\delta=\tfrac{1}{2}(\iota-\kappa)(\iota+\kappa-1),$$

an equation which establishes a relation between the four indices of the superlinear branch.

13. If we consider any term whatever in the series $\overline{\mathrm{B}}$, for example the term $(i)=\beta_i \mathfrak{b}^{\beta_i}(q-q_0)^{\frac{\beta_i}{\Delta}}$, we shall in general find a corresponding term (I) in the series H, containing $\mathrm{Q}-\mathrm{Q}_0$ raised to the power $\frac{\beta_i}{\beta_1-\Delta}$: (I) may be considered as the sum of two parts, I_1 and I_2, of which the first, I_1, arises from the term (i) itself, the other, I_2, from the terms preceding (i); (I) being in no way affected by the terms following (i). If β_i is one of the critical exponents, we have just seen that $\mathrm{I}_2=0$, $\mathrm{I}_1=\mu^{\beta_i}\,\mathrm{B}_i\,(\mathrm{Q}-\mathrm{Q}_0)^{\frac{\beta_i}{\beta_1-\Delta}}$. If β_i is not one of the critical exponents, the first of these equations ceases to subsist, but the second remains true, and its proof requires only a slight modification of the reasoning in Art. 12. Now let two series $\overline{\mathrm{B}}$, appertaining to two different superlinear branches, which have a common tangent, coincide as far as the term (i), but exclusively of it; the two corresponding series $\overline{\mathrm{H}}$ will coincide as far as the term (I), but exclusively of it; we suppose $i>0$. That all terms preceding (I) will coincide in the two developments $\overline{\mathrm{H}}$ is evident, for these terms arise solely from the terms preceding (i), which are identical in the two developments $\overline{\mathrm{B}}$. And the terms (I) themselves are different: for the difference of the two terms (i) is $(\mathrm{B}_i-\mathrm{B}_i')\,\omega^{\beta_i}(q-q_0)^{\frac{\beta_i}{\Delta}}$, where one of the two B_i, B_i' may be zero, but the difference $\mathrm{B}_i-\mathrm{B}_i'$ is by hypothesis not zero; and the difference of the two terms (I) is $\mu^{\beta_i}(\mathrm{B}_i-\mathrm{B}_i)\,\theta^{\beta_i}\times(\mathrm{Q}-\mathrm{Q}_0)^{\frac{\beta_i}{\beta-\Delta}}=\mathrm{I}_1-\mathrm{I}_1'$, for these two terms have the same part I_2.

Let $\overline{\mathrm{D}}$ be the number of points, $\overline{\mathrm{T}}$ the number of tangents common to the two branches $\overline{\mathrm{B}}$ at the point (p_0, q_0); $\overline{\mathrm{T}}$ is given by the formula

$$\overline{\mathrm{T}}=\beta_i\frac{\gamma'-\Delta'}{\sigma-d}+\frac{(\gamma-\Delta)(\gamma'-\Delta')}{(\sigma-d)^2}\{\sigma(\sigma-d-d_1)+\sigma_1(d_1-d_2)+\ldots\ldots\},$$

which is derived from the expression for N′ in Art. 8, by writing $\gamma-\Delta$ for Δ, $\gamma'-\Delta'$ for Δ', and $\sigma-d$ for d. Observing that $\frac{\gamma}{\Delta}=\frac{\gamma'}{\Delta'}=\frac{\sigma}{d}$, whence $\frac{\gamma'-\Delta'}{\sigma-d}=\frac{\Delta'}{d}$, $\frac{\gamma-\Delta}{\sigma-d}=\frac{\Delta}{d}$, $\frac{\Delta}{d}\,\sigma=\gamma=\iota+\kappa+2$,

$$\frac{\Delta'}{d}\sigma = \gamma' = \iota' + \kappa' + 2, \quad \frac{\iota+1}{\kappa+1} = \frac{\iota'+1}{\kappa'+1},$$

we find $\overline{\mathrm{T}} - \overline{\mathrm{D}} = \frac{\Delta\Delta'}{d^2}\sigma(\sigma - 2d)$

$$= (\iota-\kappa)(\iota'+\kappa'+2) = (\iota'-\kappa')(\iota+\kappa+2)$$
$$= (\iota+1)(\iota'+1) - (\kappa+1)(\kappa'+1) = \Delta_1\Delta_1' - \Delta\Delta'.$$

We have supposed in the demonstration that $i > 1$, or that the two developments of $p - p_0 - \mathrm{B}_0(q - q_0)$ coincide for at least one term. But, for the validity of the formulæ, it is only necessary that the first exponent should be the same in the two developments; and indeed the last two expressions for $\overline{\mathrm{T}} - \overline{\mathrm{D}}$ hold universally for any two superlinear branches having a common tangent.

14. The *species* of a superlinear singularity may be regarded as defined by the series of numbers Δ and Δ_1; $\Delta_1, \Delta_2, \ldots\ldots, \gamma_1, \gamma_2, \ldots\ldots$, so that two superlinear singularities, for which these indices have the same values, may be considered as belonging to the same species. A rougher classification, however, which is sometimes useful, may be obtained in the following way. Leaving out of sight the case in which two superlinear singularities present themselves as conjugate imaginaries, and attending only to the case of a real superlinearity, we may distinguish four varieties differing from one another in the appearance which they present to the eye. (See a Memoir by M. Stolz, *Mathematische Annalen*, Vol. VIII., p. 440.)

(i.) Δ uneven, Δ_1 uneven; no apparent cusp or inflexion.
(ii.) Δ even, Δ_1 uneven; an apparent cusp, no apparent inflexion.
(iii.) Δ uneven, Δ_1 even; an apparent inflexion, no apparent cusp.
(iv.) Δ even, Δ_1 even; an apparent cusp, and an apparent inflexion.

The form of (ii.) is that of the common or keratoid cusp; (iv.) has the form of the cusp of the second species, or rhamphoid cusp. There is an apparent inflexion at the rhamphoid cusp, because, if a person describing the curve continuously passes through the cusp, the concavity of the curve is to his right after he has passed through the cusp, if it was to his left hand before, and *vice versâ*. We may further observe that, in case (iv.), Δ and Δ_1, being both even, have a common measure; thus $\Delta_2 > 1$, and the superlinearity is *composite*. The cases (ii.) and (iii.) are correlative; the cases (i.) and (iv.) are their own correlatives.

15. The curvature of a curve at two points infinitely near to a given superlinear point, and at equal distances from it on either side, is always the same; and is infinite, finite, or zero, according as $\Delta > \Delta_1$, $\Delta = \Delta_1$, or $\Delta < \Delta_1$. Thus, in each of the cases (i.) and (iv.), there are

three sub-varieties of form; and two in each of the cases (iii.) and (ii.). The following are the simplest examples of each of these sub-varieties: for the sake of completeness, the cases in which either of the two numbers Δ or Δ_1 is unity, are included.

(i.) Δ and Δ_1, uneven.

$$\Delta > \Delta_1: \quad y = x^{\frac{\Delta}{\Delta_1}}; \quad y = x^{\frac{\Delta_1}{\Delta}}.$$
$$\Delta = \Delta_1: \quad y = x^2; \quad y = x^2 + x^{\frac{7}{3}}.$$
$$\Delta < \Delta_1: \quad y = x^4; \quad y = x^{\frac{\Delta_1}{\Delta}}.$$

(ii.) Δ even, Δ_1 uneven.

$$\Delta > \Delta_1: \quad y = x^{\frac{3}{2}}; \quad y = x^{\frac{7}{4}}.$$
$$\Delta < \Delta_1: \quad y = x^{\frac{5}{2}}.$$

(iii.) Δ uneven, Δ_1 even.

$$\Delta > \Delta_1: \quad y = x^{\frac{5}{2}}.$$
$$\Delta < \Delta_1: \quad y = x^3; \quad y = x^{\frac{7}{3}}.$$

(iv.) Δ and Δ_1 even.

$$\Delta > \Delta_1: \quad y = x^{\frac{3}{2}} + x^{\frac{7}{4}}.$$
$$\Delta = \Delta_1: \quad y = x^2 + x^{\frac{5}{2}}.$$
$$\Delta < \Delta_1: \quad y = x^3 + x^{\frac{7}{3}}.$$

It should be noticed that in the equation $y = x^{\frac{3}{2}} + x^{\frac{7}{4}}$, the only independent radical is $x^{\frac{1}{4}}$, and that $x^{\frac{1}{2}}$ is to be interpreted as $(x^{\frac{1}{4}})^2$. Thus, supposing x positive, and understanding by $\sqrt{x^3}$ and $\sqrt[4]{x^7}$ the real and positive values of the radicals, we have for the four partial branches the equations

$$y = \sqrt{x^3} + \sqrt[4]{x^7}, \qquad y = \sqrt{x^3} - \sqrt[4]{x^7},$$
$$y = -\sqrt{x^3} - i\sqrt[4]{x^7}, \qquad y = -\sqrt{x^3} + i\sqrt[4]{x^7},$$

of which the first two appertain to a real rhamphoid cusp. If we were to change the sign of $\sqrt{x^3}$, we should pass from the equation

$$U = (y^2 + x^3)^2 - x^3(x^2 + 2y)^2 = 0,$$

which is the rationalized equivalent of $y = x^{\frac{3}{2}} + x^{\frac{7}{4}}$, to the equation

$$V = (y^2 + x^3)^2 - x^3(x^2 - 2y)^2 = 0,$$

which is the rationalized equivalent of $y = -x^{\frac{3}{2}} + x^{\frac{7}{4}}$. It is, of course, quite possible that two developments, such as $y = \pm x^{\frac{3}{2}} + x^{\frac{7}{4}} + \ldots$, may both belong to the same curve (as indeed they do both belong to the curve $UV + x^{15}\phi(x, y) = 0$), but such a curve would have two distinct superlinear branches touching one another at the point $x=0$, $y=0$.

16. Let O be any point whatever on a curve line; let the arc $OP = \sigma$, P being a point on the curve infinitely near to O; let M be the orthogonal projection of P on the tangent at O; and let the tangents at O

and P intersect at T, making the infinitesimal angle ω. Then it will be found that $\frac{\Delta_1}{\Delta} = \frac{\log \omega}{\log \sigma} = \frac{\mathrm{OT}}{\mathrm{TP}} = \frac{\angle \mathrm{TPO}}{\angle \mathrm{TOP}} = \frac{\log \mathrm{MP}}{\log \mathrm{OM}} - 1.$

The fraction $\frac{\Delta_1}{\Delta}$ which admits of these various geometrical interpretations may perhaps be called the *logarithmic curvature* of the curve at the point O. At any ordinary point it is unity; and in a geometrical curve it is always rational, but in a transcendental curve it may have any value rational or irrational.

Since Δ or $\kappa+1$ is the number of points in which the superlinear branch is cut by any line passing through O, other than its tangent at the point O, we infer that, correlatively, $\iota+1$ or Δ_1 is the number of tangents drawn to the superlinear branch from any point on the tangent at O, other than O itself. Thus, if d be the discriminantal index of O, or the number of points in which the curve is cut at O by the polar of any arbitrary point, $d+\Delta$, is the number of points in which the curve is cut at O by the polar of any point on the tangent at O, other than O itself; there is, of course, a correlative definition of $d+\Delta$. Lastly, since $\Delta+\Delta_1$ is the number of points common at O to the tangent and the curve, it is also, correlatively, the number of tangents drawn from O to touch the curve at that point. Thus the polar of the point O intersects the curve at O in $d+\Delta+\Delta_1$ points, and the tangent at O counts as $d+\Delta+\Delta_1$ tangents common to the curve, and to the tangential polar of OT with regard to the curve. For the numbers $\Delta_1, \Delta_2, \ldots \gamma_1, \gamma_2, \ldots$ no simple geometrical definition has as yet presented itself.

17. The proof of Plücker's formulæ, which is indicated in Art. 10, may appear very indirect. Some further observations on these formulæ, and on the various modes of demonstrating them, may not be out of place.

(1.) If we write $\mathrm{D} = \Sigma(2\delta+3\kappa)$, $\mathrm{T} = \Sigma(2\tau+3\iota)$, $\mathrm{I} = \Sigma\iota$, $\mathrm{K} = \Sigma\kappa$, $\frac{1}{3}(\mathrm{I}-\mathrm{K}) = \Omega$, Plücker's formulæ become

$$\begin{aligned} n &= m(m-1)-\mathrm{D}, \\ m &= n(n-1)-\mathrm{T}; \\ \Omega &= m(m-2)-\mathrm{D}, \\ -\Omega &= n(n-2)-\mathrm{T}; \end{aligned}$$

giving

$$\mathrm{T}-\mathrm{D} = n^2-m^2, \quad \Omega = 3(n-m),$$
$$\Omega^4-2\Omega^2(\mathrm{T}+\mathrm{D})-4\Omega(\mathrm{T}-\mathrm{D})+(\mathrm{T}-\mathrm{D})^2 = 0.$$

It is thus apparent that Plücker's equations do not contain either K or I separately, but only the difference I−K.

(2.) The discriminantal index $d = 2\delta+3\kappa$ of any given point is defined geometrically as the number of intersections of the polar of an

arbitrary point with the curve at the given point. But the definition which we have given in Art. 9 of the cuspidal index κ is an analytical one, and does not readily admit of interpretation in coordinate geometry. The Hessian does not serve to define either ι or κ, for in all the cases that have as yet been rigorously investigated, it has been found that the number of intersections of the Hessian with the curve at a point of discriminantal index d is $3d+\iota-\kappa$, so that, even if the number of these intersections at any singular point should be determined by a general method, we should only obtain a definition of the difference $\iota-\kappa$. Again, if several superlinear branches have a common tangent OT at the point O, it will be seen that the geometrical definitions of Art. 16 only give the numbers $\Sigma(\iota+1)$ and $\Sigma(\kappa+1)$; viz., if d is the total discriminantal index of all the branches intersecting at O, the first polar of any point on OT (other than O) intersects the curve at O in $d+\Sigma(\iota+1)$ points; the polar of O intersects the curve at O in $d+\Sigma(\iota+\kappa+2)$ points; and there are correlative definitions of the numbers $d+\Sigma(\kappa+1)$, and $d+\Sigma(\iota+\kappa+2)$. By combining these definitions, we obtain a geometrical definition of the difference $\Sigma(\iota-\kappa)$, the summation extending to all the branches which touch one another at O. But here it is to be observed (1) that to deduce the values of $\Sigma\iota$ and $\Sigma\kappa$ from those of $\Sigma(\iota+1)$ and $\Sigma(\kappa+1)$, we should require to determine the number λ of distinct superlinear branches which touch OT at O; and (2) that, even if $\Sigma\iota$ and $\Sigma\kappa$ were known, it would still remain to determine the decomposition of these sums, and to assign the partial indices appertaining to each of the λ branches; whereas no determination of the number λ, or of the indices ι and κ of each separate superlinear branch, has as yet been obtained by considering the intersections of the given curve with any concomitant or system of concomitants.

(3.) The difficulty, which thus presents itself in obtaining a definition of the indices ι and κ, ceases to exist when we leave the domain of coordinate geometry, and consider either the analytical expansions, or the geometrical representations (depending on principles foreign to coordinate geometry) which correspond to those expansions. If several superlinear branches touch one another at a given point, the analytical expansions separate them, and assign the cuspidal and inflexional indices proper to each of them. If we apply to the equation $F(p, q) = 0$ the geometrical methods of double algebra, the cuspidal indices appear in the cycles of values of p, which present themselves at the points answering to the discriminantal values of q. (See the memoir of M. Puiseux, *Liouville*, Vol. XV., p. 384.) If, instead of the simple plane of double algebra, we use the multiple plane of Riemann, the cuspidal indices are represented by the spires which connect the leaves of the multiple plane. But it is important

to remember that, in employing the methods of double algebra, and *à fortiori* in employing the surfaces of Riemann, we are entirely abandoning the methods of coordinate and projective geometry. The present question is perhaps not directly affected by the fundamental distinction between the "infinite" of double algebra, which is a point, and the infinite of projective geometry, which is a straight line. But the duality, characteristic of projective geometry, is lost in double algebra; so that, when the complex values of p and q which satisfy the equation $F(p, q) = 0$ are regarded as developed on a plane, or on one of Riemann's surfaces, we do indeed obtain a direct representation of the cuspidal index κ, but no corresponding representation (unless we first transform the equation into its reciprocal) of the correlative index ι. Indeed, it may be asserted that, whereas the character of any given superlinearity mainly depends on a series of indices $\Delta=\kappa+1$, $\Delta_1=\iota+1$, Δ_1, Δ_2, ..., γ_1, γ_2, .. , the modes of geometrical representation, to which we are here referring, offer a sensible image of the first of these indices only. If we employ a simple plane, any one of the Δ values of p, which come to coincide with one another at the discriminantal point, must describe Δ elementary contours around that point before it acquires again its original value. If for simplicity we suppose that $\Delta_2=1$, the Δ values of p, which form the cycle, will divide themselves into Δ_1 sub-cycles, each containing $\frac{\Delta}{\Delta_1}$ values; and any value, belonging to one of these sub-cycles, will acquire approximately its original value, after describing $\frac{\Delta}{\Delta_1}$ elementary contours around the discriminantal point, the order of the error being $\frac{\gamma_1}{\Delta}$ if the order of the infinitesimal radius be taken as unity. And upon this approximate return to the original value depends the only indication which the method affords of the existence of sub-cycles, and of the values of the numbers Δ_1 and γ_1. If we employ the multiple plane of Riemann, we may perhaps represent the relations of the Δ expansions to one another by taking a Δ_1-leaved plane, repeated $\frac{\Delta}{\Delta_1}$ times, and having a spire of order $\Delta-1$, so arranged that after $\frac{\Delta}{\Delta_1}$ revolutions we return to the same Δ_1-leaved plane upon which we were when we set out, but not to the same leaf of that plane. And we can give to this image a certain amount of clearness by supposing that the Δ_1 leaves of any Δ_1-leaved plane are infinitely nearer to one another than are any two of the $\frac{\Delta}{\Delta_1}$ repetitions of the Δ_1-leaved plane.

(4.) The demonstrations of Plücker's formulæ, which are usually

given, apply only to the case in which the singularities are simple; the cases of multiple points, or multiple tangents, or of branches having contact with one another of any order, being made to depend, by the method of limits, on the simple cases of double points, or double tangents (see Dr. Salmon's *Higher Plane Curves*, p. 53). But these demonstrations do not admit of immediate extension to the case of the higher singularities properly so called, because it has not as yet been established, in any general manner, that a higher singularity may be regarded as the limit of an equivalent number of lower singularities situated infinitely near to one another. It would seem that Plücker himself was well aware of the incompleteness (in this respect) of the demonstration of his equations; for he supplements that demonstration by separately considering the case of a common cusp of the second species. Assuming the equation $n = m(m-1)-D$, and its reciprocal, (about the rigorous proof of which there is no doubt,) we have only to establish one other equation of the system. Two different methods are given by Plücker (*Theorie der Algebraischen Curven*, Part ii., Arts. 77—81): (i.) He establishes directly the theorem that, at a cusp of the second species, the curve

$$\frac{d^2F}{dp^2}\left(\frac{dF}{dq}\right)^2 - 2\,\frac{d^2F}{dp\,dq}\,\frac{dF}{dp}\,\frac{dF}{dq} + \frac{d^2F}{dq^2}\left(\frac{dF}{dp}\right)^2 = 0$$

(which may be used for our present purpose instead of the Hessian) intersects the given curve in $3d+\iota-\kappa = 15$ points. We have already stated that, in all the cases which have been examined hitherto, the number of intersections of the Hessian with the curve at any point has been found to be $3d+\iota-\kappa$; but no general demonstration of this theorem has as yet been given. The only method at present known for determining the number of intersections of two curves at a point which is singular on each of them, consists in obtaining the developments of the various branches of the two curves at the point, and in comparing these developments with one another. The discussion in Art. 18 of the development of the polar curve in the vicinity of a superlinear branch, may serve to show that the corresponding enquiry in the case of the Hessian is one of considerable intricacy. (ii.) The other method employed by Plücker depends on a determination of the number of double tangents lost by a curve of the fourth order in consequence of the presence of a cusp of the second species. In the absence of any demonstration that a higher singularity can be regarded as the limit of simple singularities existing infinitely near to one another, it is difficult to see how this mode of proof can be rendered universally applicable.

(5.) We have seen (Art. 10) that the theorem of the invariance of the number $\frac{1}{2}(m-1)(m-2)-\Sigma\delta-\Sigma\kappa$ in any unicursal transformation of

the curve suffices to establish the equation

$$(A) \ldots\ldots \tfrac{1}{2}(m-1)(m-2)-\delta-\kappa = \tfrac{1}{2}(n-1)(n-2)-r-\iota,$$

and thus to complete the proof of the formulæ of Plücker. Among the demonstrations of this theorem which have been given in recent times that of MM. Bertini and Zeuthen (*Giornale di Mathematica*, Vol. VII., p. 105; *Mathematische Annalen*, Vol. III., p. 150; Dr. Salmon's *Higher Plane Curves*, p. 314) is remarkable for its simplicity; and appears, as we shall now attempt to show, to admit of extension to the case in which the curves have any singularities whatever. We begin by assuming that when a curve is subjected to an unicursal, or one-to-one transformation, the continuity of its branches is invariably preserved, even when the position of these branches with regard to one another has undergone great distortion. For example, if a curve have two branches intersecting at the point O, these two branches will certainly be represented by two corresponding branches in the transformed curve; but these two branches may have no point of intersection, and the point O may be represented by two different points one on each of the two branches. Again, two branches which osculate one another with any degree of approximation may be transformed into branches having no contact and no point in common. But a superlinear branch behaves as one branch, and always is transformed into one branch and one only. Consider, for example, a real branch which is superlinear at O, and suppose for simplicity that no other branch passes through O; whatever be the nature of the superlinearity, we have one continuous branch passing through O, and if a point describe this branch, the track of the image-point in the transformed figure cannot be anything but one continuous branch.

Let C_1, C_2 be two curves of the orders m_1, m_2, and of the classes n_1, n_2, lying in the same plane and corresponding to one another unicursally; and let P_1, P_2 be points upon them corresponding unicursally. Taking two arbitrary points S_1, S_2, we consider, with M. Zeuthen, the locus Γ of the intersection of the rays S_1P_1, S_2P_2; and we propose to determine the number of tangents that can be drawn to Γ from each of the two points S_1 and S_2. We may suppose that S_1S_2 cuts each of the two curves in points which do not have singular points of the other curve for their corresponding points; then it is evident that Γ will have m_2 ordinary branches passing through S_1, and m_1 ordinary branches passing through S_2. We may further suppose that the n_1 tangents drawn from S_1 to C_1 are none of them singular tangents, and that to the points of contact of these n_1 tangents there answer on C_2 points having no singularity: each of these tangents will then be a tangent of Γ, but not a singular tangent of that curve. Beside the $2m_2+n_1$ tangents, which we have now drawn from S_1 to Γ, there may be others, coinciding in direction with the rays running from S_1 to the

singular points of C_1. Let X_1 be a superlinear point on C_1, having the cuspidal index κ_1; and to X_1 let X_2 answer on C_2, the cuspidal index of X_2 being κ_2, where $\kappa_2 \gtreqless 0$. We may suppose at first that only one branch passes through X_1 and only one through X_2. The ray S_1X_1 meets C_1 at X_1 in precisely κ_1+1 coincident points, because S_1X_1 is not a tangent at X_1; similarly S_2X_2 is not a tangent at X_2, but meets C_2 in precisely κ_2+1 points at X_2, since we may attribute to S_2 the requisite generality of position with regard to C_2. Thus, if Q is the intersection of S_1X_1, S_2X_2, the locus Γ is intersected at Q κ_1+1 times by S_1X_1, and κ_2+1 times by S_2X_2. The points of the curve Γ answer, one to one, to the points of C_1 or C_2; thus at Q there is but one branch answering to the one branch at X_1, or to the one branch at X_2. If $\kappa_1 = \kappa_2$, the cuspidal index of this branch is $\kappa_1 = \kappa_2$, while its inflexional index remains unknown. If $\kappa_1 > \kappa_2$, its cuspidal index is κ_2, its inflexional index is $\kappa_1-\kappa_2-1$; similarly, if $\kappa_2 > \kappa_1$, these indices are κ_1 and $\kappa_2-\kappa_1-1$; *i.e.*, in the first case, S_1X_1 counts $\kappa_1-\kappa_2$ times as a tangent to Γ at Q, and S_2X_2 is not a tangent at all; in the second case, S_2X_2 counts $\kappa_2-\kappa_1$ times as a tangent, and S_1X_1 is not a tangent at all. When $\kappa_1 = \kappa_2$, neither S_1X_1 nor S_2X_2 are tangents. The preceding reasoning will not be affected, if we now introduce the supposition that several linear or superlinear branches intersect or osculate at X_1, and that branches corresponding to some or all of them pass through X_2. Several branches will now pass through Q, but each of them may be considered separately, and the number of times that it is touched by S_1Q or S_2Q may be ascertained as above. Equating the results appertaining to the points S_1 and S_2, we now obtain

$$2m_2+n_1+\Sigma'(\kappa_1-\kappa_2) = 2m_1+n_2+\Sigma'(\kappa_2-\kappa_1);$$

where Σ' extends only to those differences which are positive. Written in the form

$$n_1+\Sigma\kappa_1-2m_1 = n_2+\Sigma\kappa_2-2m_2,$$

this equation coincides with the formula

$$\tfrac{1}{2}(m_1-1)(m_1-2)-\Sigma(\delta_1+\kappa_1) = \tfrac{1}{2}(m_2-1)(m_2-2)-\Sigma(\delta_2+\kappa_2),$$

which it was required to prove.

The assumption, which we have explicitly made, that a linear or superlinear branch is always transformed by a one-to-one transformation into one branch, and one only, is indispensable in the preceding proof; as upon it depends the determination of the number of times that Γ is touched by S_1X_1 or S_2X_2. In the case of a real branch transformed by a real transformation, the assumption may be regarded as evident; in the general case, we should have to consider, instead of two plane curves, the two corresponding surfaces of Riemann. For our immediate purpose, however, we do not need to establish the assumption as universally true in all cases; because the only one-to-one transformation (beside that of C_1 or C_2 into Γ) which is here employed

is the transformation by polar reciprocation; and the investigation of Art. 11 affords a direct proof that in this transformation any one linear or superlinear branch is always transformed into one branch (linear or superlinear).

(6.) Abandoning for a time the hypotheses of Art. 1, let us suppose that P is a singular point on the curve C, Q retaining its generality of position. And first let P be a point through which only one superlinear branch passes, having the indices $\kappa = \Delta - 1$, $\iota = \Delta_1 - 1$; let us also suppose that no singular tangent of C (other than the tangent at P) passes through P. The order of p in the equation $F(p, q) = 0$ is now $m - \Delta$, instead of m; and the number of tangents that can be drawn from P to the curve C (other than the coincident tangents at P itself) is $n - \Delta - \Delta$, (see Art. 16), instead of n. To all the singular points of C, other than P, there will appertain developments of precisely the same form as in the case in which P has no speciality of position. Let q_0 be the value of q corresponding to the tangent at P; the parameters of the point P are $p = \infty$, $q = q_0$. We cannot, therefore, in examining the superlinear branch at P, develope p in a series proceeding by powers of $q - q_0$; but we may so develope $\frac{1}{p}$, or any linear function of p, such as $\frac{c + dp}{a + bp}$, which assumes a finite value $p_0 = \frac{d}{b}$, when $p = \infty$. The exponents in any such development will have Δ_1 instead of Δ, for their least common denominator, because the tangent to C at P meets the curve (Art. 16) in $\Delta + \Delta_1$ points, so that, if $q = q_0$, Δ_1 of the $m - \Delta$ values of $\frac{c + dp}{a + bp}$ become equal to p_0. Setting out from the given equation $F(p, q) = 0$, let us form the developments appertaining to all the *singular* discriminantal values of q; and in each group of conjugate developments let us consider the greatest common divisor θ of its exponents. The sum $\Sigma(\theta - 1)$ will be equal to $\Sigma\kappa + \Delta_1 - \Delta_1$, instead of $\Sigma\kappa$; and the three numbers, by which we have now replaced m, n, and $\Sigma\kappa$, will satisfy the equation

$$(n - \Delta - \Delta_1) + (\Sigma\kappa + \Delta_1 - \Delta) - 2(m - \Delta) = n + \Sigma\kappa - 2m.$$

The cases in which (α) more than one branch passes through P, (β) one or more singular tangents pass through P, (γ) Q as well as P has some speciality of position with regard to C, may all be treated by the same method. In any of these cases, let $E(p)$ be the highest exponent of p in the equation $F(p, q) = 0$; and let $\omega(p) = \Sigma(\theta - 1)$, the sign of summation now extending to *all* the discriminantal values of q, so that $\Sigma(\theta - 1)$ contains an unit for every ordinary tangent that can be drawn from P to touch the curve elsewhere. If any of the discriminantal values of q, or any of the corresponding equal values of p, are infinite,

we are to employ linear functions of p and q, instead of p and q themselves, in forming the developments from which we are to infer the numbers θ. We shall thus obtain the equation

$$(B) \ldots\ldots\ldots \omega(p) - 2E(p) = \omega(q) - 2E(q) = n + \Sigma\kappa - 2m,$$

from which, as Clebsch has shown, the general theorem of the invariance of the deficiency may be immediately deduced. (See a Memoir by M. Nöther, *Mathematische Annalen*, Vol. VIII., p. 497.)

In the memoir to which we have just referred, M. Nöther offers a demonstration of the equation (B). But this demonstration is perhaps not wholly free from obscurity. (See the words, p. 499, *loc. cit.*, "Dieses findet . . . ergiebt," with the accompanying reference to the Göttingen *Nachrichten*.) A similar remark applies to a second demonstration, in the same memoir, of the invariance of the deficiency. [See p. 501, "Man hat aber dann . . . das Glied $\Sigma i_1(i_1-1)$."]

M. Nöther has returned to the same subject, in a recent memoir of great interest (*Mathematische Annalen*, Vol. IX., p. 166), in which he considers the resolution of a higher singularity by successive applications of a simple quadratic transformation, and infers (though by a method which can hardly be accepted as rigorous) that any higher singularity may be regarded as the limit of a certain number of lower singularities situated infinitely near to one another. We may observe (α) that the use of a quadratic transformation for the resolution of complicated singularities is due to Cramer (*Analyse des Lignes Courbes*); (β) that to establish the complete system of the formulæ of Plücker, M. Nöther selects the same three equations, which we have been led to employ in the present paper [viz., the equations (i.), (ii.), and (iii.) = (iv.), of Art. 10].

18. The expansions of Arts. 3 and 4 enable us to examine the relation of a curve at a singular point to its polar curves. Putting for brevity $p - p_0 = \eta$, $q - q_0 = \xi$, $F(p, q) = F_1(\eta, \xi)$, we have $F_1(\eta, \xi) = \Pi(\eta - \bar{A}) \times \Pi(\eta - \bar{B})$, $\frac{dF}{dp} = \frac{dF_1}{d\eta}$. From the expression of $F_1(\eta, \xi)$ as a product of m factor-series, we infer that if, on writing $K_1\xi$ for η in $F_1(\eta, \xi)$, we obtain a result of which the order of evanescence with ξ is higher than μ, $\eta = K_1\xi + \ldots\ldots$ is the beginning of one at least of the expansions $\bar{B}$. Again, let us substitute for η in $F_1(\eta, \xi)$ an expression of the form

$$K = K_1\xi + K_2\xi^{\alpha_2} + K_3\xi^{\alpha_3} + \ldots\ldots + K_\nu\xi^{\alpha_\nu},$$

in which $1 < \alpha_2 < \alpha_3 \ldots\ldots < \alpha_\nu$. If the order of evanescence of $F_1(K, \xi)$ with ξ experiences an abrupt diminution when either α_ν or K_ν (the exponent and coefficient of the last term of K) is affected by any small variation, the terms K are the initial terms of one at least of the expansions $\bar{B}$. This observation (which admits of some useful appli-

cations) enables us to deduce the developments appertaining to the polar curve $\frac{dF}{dp}$, in the vicinity of the point (p_0, q_0), from the developments appertaining to C.

Let k of the developments $\overline{B}$ coincide with one another and with K, as far as the term $K_\nu \xi^{\alpha_\nu}$ inclusively, so that for any one of these k developments we have

$$\overline{B}_i = K + L_i \xi^{l_i}, \quad l_i > \alpha_\nu \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (K),$$

$$L_i = \lambda_i + \lambda_i' \xi^{l_i'} + \ldots\ldots,$$

the terms $\lambda_i \xi^{l_i}$ not being all identical.

Put $$V = \eta - K, \qquad \prod_1^k (V - L_i \xi^{l_i}) = \phi(V);$$

then $$F_1(\eta, \xi) = M \times \phi(V), \quad \text{and} \quad \frac{dF_1}{d\eta} = \frac{dM}{d\eta}\phi(V) + M\phi'(V),$$

M being a product of $m-k$ factors, viz., of the $m-\mu$ factors $\eta - \overline{A}$, and of those $\mu - k$ factors $\eta - \overline{B}$ which do not coincide with $\eta - K$ as far as the term $K_\nu \xi^{\alpha_\nu}$ inclusively. Suppose, at first, that $l_1, l_2, \ldots\ldots l_k$ are all unequal, and arranged in order of magnitude; it is easily ascertained that the first terms in the expansions of the roots of $\phi'(V) = 0$ are

$$V_1 = \frac{k-1}{k}\lambda_1 \xi^{l_1}, \quad V_2 = \frac{k-2}{k-1}\lambda_2 \xi^{l_2}, \quad V_3 = \frac{k-3}{k-2}\lambda_3 \xi^{l_3}, \quad \ldots\ldots$$

$$V_{k-1} = \tfrac{1}{2}\lambda_{k-1} \xi^{l_{k-1}}.$$

Substitute for η in $\frac{dF_1}{d\eta}$ an expression of the form $\eta = K + H\xi^h$, where $h > \alpha_\nu$, and H is independent of ξ. If $H\xi^h$ is not the same as any one of the quantities $V_1, V_2, \ldots V_{k-1}$, the order of evanescence of $\phi(V)\frac{dM}{d\eta}$ surpasses that of $M\phi'(V)$; for the order of evanescence of M cannot surpass that of $\frac{dM}{d\eta}$ by a number greater than α_ν, whereas the order of $\phi(V)$, on the supposition that none of the equations $H\xi^h = V_i$ is satisfied, surpasses the order of $\phi'(V)$, at least by one of the numbers $l_1, l_2, \ldots l_k$. If we now suppose H and h to vary continuously, the order of evanescence of $\phi'(V)$ is abruptly increased when $H\xi^h$ comes to coincide with any one of the roots $V_0, V_1 \ldots\ldots V_{k-1}$; and, since the order of evanescence of M remains unchanged, that of $\frac{dF}{d\eta}$ is also increased abruptly. Hence $k-1$ of the developments appertaining to $\frac{dF}{d\eta}$ are of the type

$$\eta = K + \frac{k-1}{k}\lambda_1 \xi^{l_1} + \ldots, \quad \eta = K + \frac{k-2}{k-1}\lambda_2 \xi^{l_2} + \ldots,$$

$$\eta = K + \tfrac{1}{2}\lambda_{k-1} \xi^{l_{k-1}} + \ldots.$$

Again, suppose that s of the indices l are equal; let for example the s lowest indices be equal; then s roots of the equation $\phi'(V) = 0$ are of the form $H_i \xi^l + \ldots$, where $l = l_1 = l_2 \ldots = l_s$; and if

$$\psi(\theta) = (\theta - \lambda_1)(\theta - \lambda_2) \ldots\ldots (\theta - \lambda_s),$$

the s coefficients H_i are the roots of the equation

$$(k-s)\,\psi(\theta) + \theta\psi'(\theta) = 0 \ldots\ldots\ldots\ldots\ldots\ldots (\theta).$$

If the s equal indices $l_1 \ldots l_s$ are followed by another set of s' indices equal to one another and to l', l' being $> l$, put

$$(\theta - \lambda_{s+1})(\theta - \lambda_{s+2}) \ldots\ldots (\theta - \lambda_{s+s'}) = \psi_1(\theta);$$

then the equation $\phi'(V) = 0$ has s' roots of the form $H_i' \xi^{l'} + \ldots$, the coefficients H_i' being the roots of the equation

$$(k-s-s')\,\psi_1(\theta) + \theta\,\psi_1'(\theta) = 0 \ldots\ldots\ldots\ldots\ldots\ldots (\theta'),$$

and so on continually. Lastly, considering any group of equal indices l, for example the group l_{s+1}, l_{s+2} $\ldots\ldots$ $l_{s+s'}$, let σ of the corresponding coefficients λ be supposed equal (in which case σ of the developments K coincide with one another for one term at least after $K_\nu \xi^{\alpha_\nu}$); the corresponding equation (θ') will have $\sigma - 1$ roots (and no more) equal to one another and to the equal coefficients λ; so that $\sigma - 1$ of the developments appertaining to the polar will coincide, as far as the term next after $K_\nu \xi^{\alpha_\nu}$, with the σ developments appertaining to C. To carry on these $\sigma - 1$ developments until their complete separation from one another, we must repeat the preceding process as often as may be necessary, using in the first instance $K + \lambda\xi^l$ instead of K, and confining our attention to the σ developments, appertaining to C, in which $K + \lambda\xi^l$ are the initial terms.

As the roots of the equations $\psi(\theta) = 0$, $\psi_1(\theta) = 0$, ... are all different from zero, so also are the roots of the equations (θ), (θ'), ..., except when the highest index l is one of a group of equal indices. In this case, if $\psi(\theta) = \Pi(\theta - \lambda)$, the sign of multiplication extending only to those coefficients λ_i which occur in terms having the greatest exponent l, the last of the equations (θ) is of the form $\psi'(\theta) = 0$, and r of its roots may be equal to zero. When this happens, in the r polar developments corresponding to the zero roots, the terms K are not followed by a term of the form $H\xi^l$, but by a term of higher exponent. To determine this term in each of the r developments, we must use, in forming $\psi(\theta)$, not simply the quantities λ_i, but as many terms of the series $\lambda_i + \lambda_i' \xi^{l_i'} + \ldots$ as may be necessary. The zero roots of $\psi'(\theta) = 0$ are then replaced by roots of the form $H\xi^a$, a being positive, and the initial terms of the r polar developments are given by the formula $K + H\xi^{l+a}$.

We shall employ the preceding method to examine the nature of the polar branches in the vicinity of a superlinear branch. We suppose

the superlinear branch to be of the type

$$[\Delta, \Delta_1, \Delta_2, \ldots \Delta_s, \Delta_{s+1} = 1; \ \gamma_1, \gamma_2, \ldots \gamma_s];$$

and we consider only the case in which this superlinear branch (Δ) is not touched by any other branch. The polar has $\Delta - 1$ branches (Δ') touching the superlinear branch. Their developments coincide with one another, and with those of (Δ), as far as the term $[x^{\frac{\gamma}{\Delta}}]$ exclusively. But at this term $\frac{\Delta}{\Delta_1} - 1$ of them cease to osculate any branch of (Δ); they do not contain the term $[x^{\frac{\gamma}{\Delta}}]$, which is replaced in each of them by a term of higher exponent, yet so that the aggregate of the $\frac{\Delta}{\Delta_1} - 1$ exponents cannot exceed $\frac{\gamma}{\Delta_1} - 1$. The remaining $\frac{\Delta}{\Delta_1}(\Delta_1 - 1)$ branches divide themselves into $\frac{\Delta}{\Delta_1}$ groups of $\Delta_1 - 1$ each. The $\Delta_1 - 1$ branches of each group are identical with one another, and with Δ_1 of the branches (Δ), as far as the term $[x^{\frac{\gamma_1}{\Delta}}]$ exclusively. At this term $\frac{\Delta_1}{\Delta_2} - 1$ branches out of each group cease to osculate any branch of (Δ), and the remaining $\frac{\Delta_1}{\Delta_2}(\Delta_2 - 1)$ divide themselves, in the same way as before, into $\frac{\Delta_1}{\Delta_2}$ groups of $\Delta_2 - 1$ each; the branches of each group being identical with one another, and with Δ_2 of the branches of (Δ), as far as the term $(x^{\frac{\gamma_2}{\Delta}})$ exclusively. In this way we obtain the following theorem in which i is to have every value from 0 to s, both inclusively.

"The polar curve of an arbitrary point has $\frac{\Delta}{\Delta_{i+1}} - \frac{\Delta}{\Delta_i}$ branches which form $\frac{\Delta_i}{\Delta_{i+1}} - 1$ superlinear branches of the type.

$$\left[\Delta' = \frac{\Delta}{\Delta_i}, \quad \Delta_1' = \frac{\Delta_1}{\Delta_i}, \ \ldots\ldots \ \Delta_{i-1}' = \frac{\Delta_{i-1}}{\Delta_i}, \quad \Delta_i' - 1;\right.$$

$$\left.\gamma' = \frac{\gamma}{\Delta_i}, \quad \gamma_1' = \frac{\gamma_1}{\Delta_i}, \ \ldots\ldots \ \gamma_{i-1}' = \frac{\gamma_{i-1}}{\Delta_i}\right].$$

These superlinear branches coincide with one another, and with the branches of (Δ) as far as the term $[x^{\frac{\gamma_i}{\Delta}}]$ exclusively; instead of the term $[x^{\frac{\gamma_i}{\Delta}}]$ each of them contains a term of higher exponent; the $\frac{\Delta_i}{\Delta_{i+1}} - 1$ superlinear branches may, but do not necessarily, group themselves into higher superlinear branches."

19. The development appertaining to a superlinear branch can always be obtained from the equation of the curve by successive applications of the "analytical triangle." The process has been described by M. Puiseux in his important memoir "Recherches sur les fonctions algébriques." (*Liouville*, Vol. XV., p. 384; see also a paper by M. de la Gournerie, *ibid.*, 2nd series, Vol. XIV., p. 425, Vol. XV., p. 1.) We propose to conclude the present paper by showing how the numbers γ, γ_1, ... Δ, Δ_1, ... present themselves in the course of the operation. Putting, as in Art. 18, η for $p-p_0$, ξ for $q-q_0$, we first of all write the equation $F_1(\eta, \xi) = 0$ in the form $u_\mu + u_{\mu+1} + \ldots$, where u_μ is a homogeneous function of ξ and η of the order μ, which is that of the singular point. If $(\eta - B_0\xi)^\alpha$ is a multiple factor of u_μ the line $\eta - B_0\xi$ is touched by branches (linear or superlinear) of which the aggregate order is α. Put $\eta - B_0\xi = v$; the resulting equation between v and ξ will give precisely α values of v in which the order of v surpasses that of ξ. Form, by the analytical triangle, the equations (of the aggregate order α in v) which give the initial terms of the expansions of these α values. These equations are of the type

$$(v^\nu - K\xi^\lambda)^{\alpha_1} = 0,$$

where λ and ν are relatively prime, $\lambda > \nu$, and $\Sigma\alpha_1\nu = \alpha$; they are always obtained linearly, except when there are s of them in which the numbers α_1, λ, ν are all the same; in which case the analytical triangle determines an equation, of order s, having constant coefficients, of which the roots are the s quantities K. There are four cases to be considered: (i.) $\alpha_1=1$, $\nu=1$; (ii.) $\alpha_1=1$, $\nu>1$; (iii.) $\alpha_1>1$, $\nu=1$; (iv.) $\alpha_1>1$, $\nu>1$. (i.) To the equation $v - K\xi^\lambda = 0$, $\lambda > 1$, answers a linear branch which, considered by itself, has no point-singularity (if λ is > 2, it is an inflexion). (ii.) To the equation $v^\nu - K\xi^\lambda = 0$ answers a superlinear branch of which the character is defined by the equations $\Delta=\nu$, $\Delta_1=1$, $\gamma=\lambda$; its development proceeds by integral powers of $\xi^{\frac{1}{\nu}}$, and the successive terms are obtained linearly by the analytical triangle. (iii.) To the equation $(v - K\xi^\lambda)^{\alpha_1} = 0$ answer α_1 branches, which may be all linear, but which also may group themselves in whole or in part into superlinear branches; if Δ, Δ', Δ'' ... are the orders of these separate linear or superlinear branches, we have $\Sigma\Delta=\alpha_1$. (iv.) To the equation $(v^\nu - K\xi^\lambda)^{\alpha_1} = 0$ answer $\alpha_1\nu$ branches, which may belong to α_1 distinct superlinear branches of the type ($\Delta = \nu$, $\Delta_1 = 1$, $\gamma=\lambda$); these superlinear branches may however themselves be grouped, wholly or in part, into branches of higher superlinearity; if Δ, Δ', Δ'' ... are the orders of the distinct superlinear branches, these numbers are all divisible by ν, and $\Sigma\frac{\Delta}{\nu} = \alpha_1$; we have also for every one of them

$\frac{\Delta}{\Delta_1} = \nu$, $\frac{\gamma}{\Delta_1} = \lambda$, Δ_1 having to be determined subsequently for each of them separately. With the cases (i.) and (ii.) we have nothing further to do; the case (iii.) may be regarded as included under (iv.); we therefore continue the process in this last case only. Put $v - K^{\frac{1}{\nu}} \xi^{\frac{\lambda}{\nu}} = v_1$, $\xi^{\frac{1}{\nu}}$ representing any one determinate value of the radical; and form by the analytical triangle equations of the type

$$(v_1^{\nu_1} - K_1 \xi^{\frac{\lambda_1}{\nu}})^{a_2} = 0,$$

of which the aggregate order in v_1 is a_1, and which give the initial terms of those a_1 values of v_1, of which the order surpasses that of $\xi^{\frac{\lambda}{\nu}}$; we have of course $\frac{\lambda_1}{\nu\nu_1} > \frac{\lambda}{\nu}$, or $\frac{\lambda_1}{\nu_1} > \lambda$; λ_1 and ν_1 are relatively prime, but we observe that λ_1 is not necessarily prime to ν. We consider the same four cases as before. (i.) To the equation $v_1 - K_1 \xi^{\frac{\lambda_1}{\nu}} = 0$, or more properly to the ν equations comprehended in it, answers a superlinear branch of the type ($\Delta = \nu$, $\Delta_1 = 1$, $\gamma = \nu$). (ii.) To the ν equations $v_1^{\nu_1} - K_1 \xi^{\frac{\lambda_1}{\nu}} = 0$ there also answers a single superlinear branch for which $\Delta = \nu\nu_1$, $\Delta_1 = \nu_1$, $\Delta_2 = 1$; $\gamma = \lambda\nu_1$, $\gamma_1 = \lambda_1$; *i.e.*, a superlinear branch of the type $\left(\frac{\Delta}{\Delta_1} = \nu, \frac{\Delta_1}{\Delta_2} = \nu_1, \Delta_2 = 1; \gamma = \lambda\Delta_1, \gamma_1 = \lambda_1\Delta_2\right)$. In this case, as well as in (i.), the discussion of the superlinearity is complete. (iii.) To the ν equations $(v_1 - K_1 \xi^{\frac{\lambda}{\nu}})^{a_2} = 0$ there may answer a_2 superlinear branches of the type $\left(\frac{\Delta}{\Delta_1} = \nu, \Delta_1 = 1; \gamma = \lambda\Delta_1\right)$; or these may group themselves in any manner into higher superlinear branches for each of which $\frac{\Delta}{\Delta_1} = \nu$, $\gamma = \lambda\Delta_1$; the numbers Δ_1 (which have to be determined for each branch separately), satisfying the condition $\Sigma\Delta_1 = a_2$. (iv.) To the equations $(v_1^{\nu_1} - K_1 \xi^{\frac{\lambda_1}{\nu}})^{a_2}$ answer a certain number of superlinear branches, for each of which $\frac{\Delta}{\Delta_1} = \nu$, $\frac{\Delta_1}{\Delta_2} = \nu_1$; $\gamma = \lambda\Delta_1$, $\gamma_1 = \lambda_1\Delta_2$; while Δ_2 and the subsequent numbers of the series have still to be determined, and may be different for each of them; we have however the equation $\Sigma\Delta_2 = \Sigma\frac{\Delta_1}{\nu_1} = a_2$. The process, which we need not follow further, may be considered to terminate for any particular development, when that development is separated from every other, and can be continued linearly. This will happen when, in the series

$a\,a_1\,a_2\ldots$, we arrive at a term equal to unity. And we shall eventually arrive at such a term; for, though the second of two consecutive indices a may be as great as the first (the equation $(v-\mathrm{K}\,\xi^{\underset{\lambda}{\nu}})^{a_1}$ may, for example, at the next step in the process, lead to only one equation; and this may be of the type $(v_1-\mathrm{K}_1\,\xi^{\underset{\lambda_1}{\nu}})^{a_1}=0$, so that we should have $a_2=a_1$), yet it is impossible for two branches to osculate one another indefinitely, because the discriminantal index is necessarily finite. If a_s be the first of the indices a which is equal to unity, we have

$$\frac{\Delta}{\Delta_1}=\nu,\quad \frac{\Delta_1}{\Delta_2}=\nu_1,\ \ldots\ \frac{\Delta_s}{\Delta_{s+1}}=\nu_s,\quad \Delta_{s+1}=1;$$

and the development appertaining to the superlinear branch is of the type

$$(\Delta=\nu\,\nu_1\,\nu_2\ldots\nu_s,\quad \Delta_1=\nu_1\,\nu_2\ldots\nu_s,\ \ldots\ldots,\quad \Delta_s=\nu_s,\quad \Delta_{s+1}=1,$$
$$\gamma=\lambda\Delta_1,\quad \gamma_1=\lambda_1\Delta_2,\ \ldots\ldots,\quad \gamma_{s-1}=\lambda_{s-1}\Delta_s,\quad \gamma_s=\lambda_s).$$

On Hamilton's Characteristic Function for a Narrow Beam of Light.

By J. Clerk-Maxwell, M.A., F.R.S.

[*Read January 8th*, 1874.]

Hamilton's characteristic function V is an expression for the time of propagation of light from the point whose coordinates are x_1, y_1, z_1 to the point whose coordinates are x_2, y_2, z_2. It is a function of these six coordinates of the two points. The axes to which the coordinates are referred may be different for the two points.

In isotropic media the differential equation of V may be written

$$\left(\frac{d\mathrm{V}}{dx}\right)^2+\left(\frac{d\mathrm{V}}{dy}\right)^2+\left(\frac{d\mathrm{V}}{dz}\right)^2=\mu^2 \quad \ldots\ldots\ldots\ldots\ldots (1),$$

where μ is the slowness of propagation at a point in the medium whose coordinates are x, y, z, and is a function of these coordinates. If the time of propagation through the unit of length in vacuum be taken as the unit of time, then μ is the index of refraction of the medium.

The form of the equation in doubly refracting media, as given by Hamilton, is not required for our present purpose.

Let OPQR be the path of a ray of light. Let the part OP be in a homogeneous medium whose index of

refraction is μ_1, and let QR be in a homogeneous medium whose index of refraction is μ_2. Between P and Q the ray may pass through any combination of media, singly or doubly refracting.

Let us consider the characteristic function from a point near to CP in the first medium to a point near to QR in the second.

Let the position of the first point be referred to rectangular axes, the origin of which is at P, and the axis of z_1 drawn in the direction PO. The axes of x_1 and y_1 may be turned at pleasure round that of z_1 into the position most suitable for our calculations.

Let the position of the second point be referred to Q as origin, to QR as axis of z_2, and to axes of x_2 and y_2 the position of which is of course independent of that chosen for x_1, y_1.

Let the ray from the first point O′ (x_1, y_1, z_1) to the second point R′ (x_2, y_2, z_2) pass through P′ $(\xi_1, \eta_1, 0)$, and Q′ $(\xi_2, \eta_2, 0)$.

We have then $$V_{O'R'} = V_{O'P'} + V_{P'Q'} + V_{Q'R'} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Here $$V_{O'P'} = \mu_1 O'P' = \mu_1 \sqrt{(x_1-\xi_1)^2+(y_1-\eta_1)^2+z_1^2} \quad \ldots\ldots\ldots (3),$$

and $$V_{Q'R'} = \mu_2 Q'R' = \mu_2 \sqrt{(x_2-\xi_2)^2+(y_2-\eta_2)^2+z_2^2} \quad \ldots\ldots\ldots (4).$$

Also
$$\begin{aligned} V_{P'Q'} = {} & V_{PQ} + \xi_1 \frac{dV}{d\xi_1} + \eta_1 \frac{dV}{d\eta_1} + \xi_2 \frac{dV}{d\xi_2} + \eta_2 \frac{dV}{d\eta_2} \\ & + \tfrac{1}{2}\xi_1^2 \frac{d^2V}{d\xi_1^2} + \xi_1\eta_1 \frac{d^2V}{d\xi_1\, d\eta_1} + \tfrac{1}{2}\eta_1^2 \frac{d^2V}{d\eta_1^2} \\ & + \tfrac{1}{2}\xi_2^2 \frac{d^2V}{d\xi_2^2} + \xi_2\eta_2 \frac{d^2V}{d\xi_2\, d\eta_2} + \tfrac{1}{2}\eta_2^2 \frac{d^2V}{d\eta_2^2} \\ & + \xi_1\xi_2 \frac{d^2V}{d\xi_1\, d\xi_2} + \xi_1\eta_2 \frac{d^2V}{d\xi_1\, d\eta_2} + \eta_1\xi_2 \frac{d^2V}{d\eta_1\, d\xi_2} + \eta_1\eta_2 \frac{d^2V}{d\eta_1\, d\eta_2} \end{aligned}$$

+terms involving higher powers and products of ξ_1, η_1, ξ_2, η_2 ..... (5).

Writing, for the sake of brevity, single symbols for the differential coefficients, we obtain for the value of $V_{O'R'}$ up to terms of the second degree inclusive

$$\begin{aligned} V_{O'R'} = {} & \mu_1 z_1 + \frac{\mu_1}{2z_1}\{(x_1-\xi_1)^2+(y_1-\eta_1)^2\} \\ & + \mu_2 z_2 + \frac{\mu_2}{2z_2}\{(x_2-\xi_2)^2+(y_2-\eta_2)^2\} \\ & + V_{PQ} + f_1\xi_1 + g_1\eta_1 + f_2\xi_2 + g_2\eta_2 \\ & + \tfrac{1}{2}a_1\xi_1^2 + c_1\xi_1\eta_1 + \tfrac{1}{2}b_1\eta_1^2 \\ & + p\xi_1\xi_2 + q\xi_1\eta_2 + r\eta_1\xi_2 + s\eta_1\eta_2 \\ & + \tfrac{1}{2}a_2\xi_2^2 + c_2\xi_2\eta_2 + \tfrac{1}{2}b_2\eta_2^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (6). \end{aligned}$$

This is the value of $V_{O'P'}+V_{P'Q'}+V_{Q'R'}$, supposing the course of the ray to be broken at P′ and Q′ in an arbitrary manner.

For the actual course of the ray the value of $V_{O'R'}$ must be stationary as regards variations of $\xi_1, \eta_1, \xi_2, \eta_2$. Hence, differentiating with respect to these variables, we obtain the four equations

$$\left.\begin{aligned}
\left(a_1+\frac{\mu_1}{z_1}\right)\xi_1 + c_1\eta_1 + p\xi_2 + q\eta_2 &= \mu_1\frac{x_1}{z_1}-f_1\\
c_1\xi_1 + \left(b_1+\frac{\mu_1}{z_1}\right)\eta_1 + r\xi_2 + s\eta_2 &= \mu_1\frac{y_1}{z_1}-g_1\\
p\xi_1 + r\eta_1 + \left(a_2+\frac{\mu_2}{z_2}\right)\xi_2 + c_2\eta_2 &= \mu_2\frac{x_2}{z_2}-f_2\\
q\xi_1 + s\eta_1 + c_2\xi_2 + \left(b_2+\frac{\mu_2}{z_2}\right)\eta_2 &= \mu_2\frac{y_2}{z_2}-g_2
\end{aligned}\right\} \quad \ldots\ldots\ldots (7).$$

Since OPQR is a ray of the system, the coordinates x_1, y_1; ξ_1, η_1; ξ_2, η_2; x_2, y_2, vanish together. Hence $f_1=g_1=f_2=g_2=0$, and if we write Δ for the determinant

$$\begin{vmatrix}
a_1+\frac{\mu_1}{z_1} & c_1 & p & q\\
c_1 & b_1+\frac{\mu_1}{z_1} & r & s\\
p & r & a_2+\frac{\mu_2}{z_2} & c_2\\
q & s & c_2 & b_2+\frac{\mu_2}{z_2}
\end{vmatrix} \quad \ldots\ldots\ldots\ldots (8).$$

Then

$$\left.\begin{aligned}
\Delta\xi_1 &= \frac{\mu_1}{z_1}\left(x_1\frac{d\Delta}{da_1}+\tfrac{1}{2}y_1\frac{d\Delta}{dc_1}\right)+\frac{1}{2}\frac{\mu_2}{z_2}\left(x_2\frac{d\Delta}{dp}+y_2\frac{d\Delta}{dq}\right)\\
\Delta\eta_1 &= \frac{\mu_1}{z_1}\left(\tfrac{1}{2}x_1\frac{d\Delta}{dc_1}+y_1\frac{d\Delta}{db_1}\right)+\frac{1}{2}\frac{\mu_2}{z_2}\left(x_2\frac{d\Delta}{dr}+y_2\frac{d\Delta}{ds}\right)\\
\Delta\xi_2 &= \frac{1}{2}\frac{\mu_1}{z_1}\left(x_1\frac{d\Delta}{dp}+y_1\frac{d\Delta}{dr}\right)+\frac{\mu_2}{z_2}\left(x_2\frac{d\Delta}{da_2}+\tfrac{1}{2}y_2\frac{d\Delta}{dc_2}\right)\\
\Delta\eta_2 &= \frac{1}{2}\frac{\mu_1}{z_1}\left(x_1\frac{d\Delta}{dq}+y_1\frac{d\Delta}{ds}\right)+\frac{\mu_2}{z_2}\left(\tfrac{1}{2}x_2\frac{d\Delta}{dc_2}+y_2\frac{d\Delta}{db_2}\right)
\end{aligned}\right\} \quad \ldots\ldots (9).$$

Substituting in (6), we obtain

$$\begin{aligned}
V_{O'R'} = V_{OR} &+\mu_1 z_1+\mu_2 z_2+\tfrac{1}{2}\mathfrak{A}_1x_1^2+\mathfrak{C}_1x_1y_1+\tfrac{1}{2}\mathfrak{B}_1y_1^2\\
&+\mathfrak{P}x_1x_2+\mathfrak{Q}x_1y_2+\mathfrak{R}y_1x_2+\mathfrak{S}y_1y_2\\
&+\tfrac{1}{2}\mathfrak{A}_2x_2^2+\mathfrak{C}_2x_2y_2+\tfrac{1}{2}\mathfrak{B}_2y_2^2 \quad \ldots\ldots\ldots\ldots\ldots (10);
\end{aligned}$$

where $$\mathfrak{A}_1 = \frac{\mu_1}{z_1} - \frac{\mu_1^2}{z_1^2\Delta}\frac{d\Delta}{da_1}, \quad \mathfrak{C}_1 = -\frac{\mu_1^2}{2z_1^2\Delta}\frac{d\Delta}{dc_1}, \quad \mathfrak{B}_1 = \frac{\mu_1}{z_1} - \frac{\mu_1^2}{z_1^2\Delta}\frac{d\Delta}{db_1},$$

$$\mathfrak{P} = -\frac{\mu_1\mu_2}{2z_1z_2\Delta}\frac{d\Delta}{dp}, \quad \mathfrak{Q} = -\frac{\mu_1\mu_2}{2z_1z_2\Delta}\frac{d\Delta}{dq},$$

$$\mathfrak{R} = -\frac{\mu_1\mu_2}{2z_1z_2\Delta}\frac{d\Delta}{dr}, \quad \mathfrak{S} = -\frac{\mu_1\mu_2}{2z_1z_2\Delta}\frac{d\Delta}{ds},$$

$$\mathfrak{A}_2 = \frac{\mu_2}{z_2} - \frac{\mu_2^2}{z_2^2\Delta}\frac{d\Delta}{da_2}, \quad \mathfrak{C}_2 = -\frac{\mu_2^2}{2z_2^2\Delta}\frac{d\Delta}{dc_2}, \quad \mathfrak{B}_2 = \frac{\mu_2}{z_2} - \frac{\mu_2^2}{z_2^2\Delta}\frac{d\Delta}{db_2} \quad \ldots\ldots (11).$$

This is the most general form of Hamilton's characteristic function for a pair of points, each of which is near the principal ray. It is a homogeneous function of the second degree in x_1, y_1, x_2, y_2, the coefficients of which are functions of z_1 and z_2.

By turning the axes of x_1 and y_1 about z_1, and those of x_2 and y_2 about z_2, we may get rid of two of the ten terms, and so reduce the expression to eight.

We may, for example, get rid of c_1 and c_2, and so of $\mathfrak{C}_1$ and $\mathfrak{C}_2$; but since, in the theory of pencils having two focal lines, terms may enter which must be added to $\mathfrak{C}_1$ and $\mathfrak{C}_2$, this transformation is not of much use.

It is better to begin by getting rid of q and r, by turning x_1 and y_1 round z_1, through an angle θ_1, such that

$$\tfrac{1}{2}\tan 2\theta_1 = \frac{pr+qs}{p^2+q^2-r^2-s^2} \quad \ldots\ldots (12),$$

and also turning x_2 and y_2 round z_2, through an angle θ_2, such that

$$\tfrac{1}{2}\tan 2\theta_2 = \frac{pq+rs}{p^2+r^2-q^2-s^2} \quad \ldots\ldots (13).$$

For these new axes the values of q and r are reduced to zero.

As an instance of the use of the characteristic function, let us find the form of the emergent pencil when that of the incident pencil is given.

The general form of the characteristic function of a pencil, whose axis is the axis of z, is

$$\mathrm{V}_1 = \mathrm{K} + \mu\left(z - \frac{x^2}{2\mathrm{A}} - \frac{xy}{2\mathrm{C}} - \frac{y^2}{2\mathrm{B}}\right) \quad \ldots\ldots (14),$$

as in the "Proceedings of the London Mathematical Society," vol. iv., p. 337 (1873); where A, B, C are lines from which, by the construction there given, a and b, the coordinates of the focal lines, and ϕ, the angle which the line a makes with the plane of xz, may be deduced.

The quantities a_1, &c., which occur as the coefficients of the characteristic function, are the reciprocals of lines.

$$\left.\begin{aligned} \text{Let}\quad \alpha_1 &= a_1 + \frac{1}{A_1}, \quad \beta_1 = b_1 + \frac{1}{B_1}, \quad \gamma_1 = c_1 + \frac{1}{C_1} \\ \alpha_2 &= a_2 + \frac{1}{A_2}, \quad \beta_2 = b_2 + \frac{1}{B_2}, \quad \gamma_2 = c_2 + \frac{1}{C_2} \end{aligned}\right\} \quad \ldots\ldots (15).$$

$$\text{also let}\quad \alpha_1\beta_1 - \gamma_1^2 = \delta_1, \quad \text{and} \quad \alpha_2\beta_2 - \gamma_2^2 = \delta_2$$

The condition that the incident pencil defined by A_1, B_1, C_1 should be conjugate to the emergent pencil defined by A_2, B_2, C_2, is that the value of the characteristic function must be the same for all rays of the pencil. Now a particular ray of the pencil may be defined either by the coordinates x_1, y_1, z_1 of a point through which it passes in the first medium, or by x_2, y_2, z_2 those of a point through which it passes in the second medium. In the first case, the coefficients of x_1^2, $x_1 y_1$, and y_1^2 will vanish, and in the second those x_2^2, $x_2 y_2$, and y_2^2. If the one set of conditions are fulfilled, the other set will be fulfilled also. Hence we may write the conditions either in the form

$$\frac{d\Delta}{d\alpha_1} = 0, \quad \frac{d\Delta}{d\beta_1} = 0, \quad \frac{d\Delta}{d\gamma_1} = 0 \quad \ldots\ldots\ldots\ldots\ldots (16),$$

or in the form

$$\frac{d\Delta}{d\alpha_2} = 0, \quad \frac{d\Delta}{d\beta_2} = 0, \quad \frac{d\Delta}{d\gamma_2} = 0 \quad \ldots\ldots\ldots\ldots\ldots (17),$$

where

$$\Delta = \begin{vmatrix} \alpha_1 & \gamma_1 & p & q \\ \gamma_1 & \beta_1 & r & s \\ p & r & \alpha_2 & \gamma_2 \\ q & s & \gamma_2 & \beta_2 \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots (18)$$

$$\begin{aligned} = {} & \delta_1\delta_2 - \alpha_1\alpha_2 s^2 + 2\alpha_1\gamma_2 rs - \alpha_1\beta_2 r^2 \\ & + 2\gamma_1\alpha_2 qs - 2\gamma_1\gamma_2(ps+qr) + 2\gamma_1\beta_2 pr \\ & - \beta_1\alpha_2 q^2 - 2\beta_1\gamma_2 pq - \beta_1\beta_2 p^2 + (ps-qr)^2 \quad \ldots\ldots\ldots (19). \end{aligned}$$

The conditions of conjugate pencils may therefore be written, either

$$\left.\begin{aligned} \alpha_2\delta_1 &= \beta_1 p^2 - 2\gamma_1 pr + \alpha_1 r^2 \\ \gamma_2\delta_1 &= \beta_1 pq - \gamma_1(ps+qr) + \alpha_1 rs \\ \beta_2\delta_1 &= \beta_1 q^2 - 2\gamma_1 qs + \alpha_1 s^2 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (20),$$

or in a form derived from this by exchanging the suffixes $_1$ and $_2$.

If the axes of coordinates are turned so that q and r vanish,

$$\Delta = \delta_1\delta_2 - \alpha_1\alpha_2 s^2 - 2\gamma_1\gamma_2 ps - \beta_1\beta_2 p^2 + p^2 s^2 \quad \ldots\ldots\ldots (21),$$

and

$$\left.\begin{aligned} \alpha_2\delta_1 &= \beta_1 p^2 \\ \gamma_2\delta_1 &= -\gamma_1 ps \\ \beta_2\delta_1 &= \alpha_1 s^2 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (22).$$

If we write

$$\left.\begin{aligned} \alpha_1 &= X_1 p, \quad \beta_1 = Y_1 s, \quad \gamma_1 = Z_1\sqrt{ps} \\ \alpha_2 &= X_2 p, \quad \beta_2 = Y_2 s, \quad \gamma_2 = Z_2\sqrt{ps} \end{aligned}\right\} \quad \ldots\ldots\ldots (23),$$

then X_1, Y_1, Z_1 will be inverse to X_2, Y_2, Z_2, and will satisfy the equations

$$\left.\begin{aligned} X_1X_2+Z_1Z_2 &= 1, \quad Z_1X_2+Y_1Z_2 = 0 \\ Z_1Z_2+Y_1Y_2 &= 1, \quad X_1Z_2+Z_1Y_2 = 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots (24).$$

Fig 2.

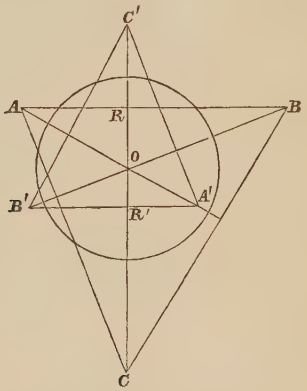
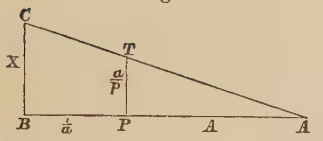


Fig 3.



The relations between the quantities X_1, Y_1, Z_1 and X_2, Y_2, Z_2, are shown in the annexed figure (Fig. 2).

Let $AR=X_1$ and $RB=Y_1$ in the same straight line, and $RO=Z_1$ perpendicular to AB. With O as centre and unity as radius describe a circle.

Draw C'B' the polar of A, and C'A' the polar of B, with respect to this circle. These lines meet BO and AO in B' and A' respectively. Join A'B' cutting RO in R'. Then $B'R'=X_2$, $R'A'=Y_2$, and $OR'=Z_2$.

Since OR' is measured downwards, Z_2 is negative in the case represented by the figure. It is manifest that X_1, Y_1, and Z_1 may be found from X_2, Y_2, Z_2, by the same process.

The relation between the quantities A, B, C and X, Y, Z is shown in Fig 3.

Let a be one of the first three or the last three of the ten coefficients of the characteristic function $V(P'Q')\ V_{P'Q'}$.

Since a is the reciprocal of a line, let BP represent the line $\frac{1}{a}$.

Draw PT perpendicular to BP so that PT is to the unit line as a to p.

Then, if PA=A, the line ATC will cut off from the line BC perpendicular to BA a part BC equal to X. For

$$BC = \frac{a}{p}+\frac{1}{A.p} = \frac{a}{p} = X.$$

In this way X may be found when A is known, or A when X is known. The same method gives the relations between B and Y and between C and Z.

The geometrical process for finding the focal lines of the emergent pencil, when those of the incident pencil are given, is therefore as follows :—

From the distances a_1 and b_1 of the focal lines, and the angle ϕ_1 between the first of them and the plane of $x_1 z_1$, deduce, by the method given in a former communication (Vol. IV., p. 337), A_1, B_1, and C_1.

From A_1, B_1, C_1, find, by the construction of Fig. 3, X_1, Y_1, and Z_1.

From X_1, Y_1, Z_1, find, by the construction of Fig. 2, X_2, Y_2, and Z_2.

From X_2, Y_2, Z_2, find A_2, B_2, C_2.

From A_2, B_2, C_2, find a_2, b_2, ϕ_2.

Thus far we have been considering the most general case of a pencil passing through any number of media between P and Q, through surfaces of any form, the media before incidence at P and after emergence at Q being isotropic. When some of these media are doubly refracting, there may be two or more emergent pencils corresponding to one incident pencil. Our investigation is applicable only to one of these emergent pencils at a time. Each emergent pencil must be treated separately.

In certain cases of practical importance the characteristic function may be greatly simplified. For instance, when the axis ray is refracted in one plane through the prisms of a spectroscope, the same positions of the axes of x and y which make q and r vanish, also make c_1 and c_2 vanish. The determinant Δ may now be written as the product of two factors

$$\Delta = \left\{ \left(a_1+\frac{\mu_1}{z_1}\right)\left(a_2+\frac{\mu_2}{z_2}\right) - p^2 \right\} \left\{ \left(b_1+\frac{\mu_1}{z_1}\right)\left(b_2+\frac{\mu_2}{z_2}\right) - s^2 \right\} \quad \ldots (25),$$

$$\begin{aligned} \mathrm{V} = \mathrm{V}_{\mathrm{PQ}} &+ \mu_1 z_1 + \mu_2 z_2 \\ &+ \frac{\frac{\mu_1}{z_1}\left\{ a_1\left(a_2+\frac{\mu_2}{z_2}\right) - p^2 \right\} x_1^2 + \frac{\mu_1\mu_2}{z_1 z_2} p x_1 x_2 + \frac{\mu_2}{z_2}\left\{ a_2\left(a_1+\frac{\mu_1}{z_1}\right) - p^2 \right\} x_2^2}{\left(a_1+\frac{\mu_1}{z_1}\right)\left(a_2+\frac{\mu_2}{z_2}\right) - p^2} \\ &+ \frac{\frac{\mu_1}{z_1}\left\{ b_1\left(b_2+\frac{\mu_2}{z_2}\right) - s^2 \right\} y_1^2 + \frac{\mu_1\mu_2}{z_1 z_2} s\, y_1 y_2 + \frac{\mu_2}{z_2}\left\{ b_2\left(b_1+\frac{\mu_1}{z_1}\right) + s^2 \right\} y_2^2}{\left(b_1+\frac{\mu_1}{z_1}\right)\left(b_2+\frac{\mu_2}{z_2}\right) - s^2} \end{aligned} \quad \ldots\ldots (26).$$

If we write

$$\left.\begin{aligned} u_1 &= \frac{\mu_1 a_2}{p^2 - a_1 a_2}, & u_2 &= \frac{\mu_2 a_1}{p^2 - a_1 a_2} \\ v_1 &= \frac{\mu_1 b_2}{s^2 - b_1 b_2}, & v_2 &= \frac{\mu_2 b_1}{s^2 - b_1 b_2} \\ f_1 &= -\frac{\mu_1 p}{p^2 - a_1 a_2}, & f_2 &= -\frac{\mu_2 p}{p^2 - a_1 a_2} \\ g_1 &= -\frac{\mu_1 s}{s^2 - b_1 b_2}, & g_2 &= -\frac{\mu_2 s}{s^2 - b_1 b_2} \end{aligned}\right\} \quad \ldots\ldots (27),$$

the characteristic function becomes

$$V = V_{PQ} + \mu_1 z_1 + \mu_2 z_2 + \frac{1}{2}\frac{\mu_1(z_2-u_2)x_1^2 + \mu_2(z_1-u_1)x_2^2 - (f_1\mu_2+f_2\mu_1)x_1x_2}{(z_1-u_1)(z_2-u_2)-f_1f_2} + \frac{1}{2}\frac{\mu_1(z_2-v_2)y_1^2 + \mu_2(z_1-v_1)y_2^2 - (g_1\mu_2+g_2\mu_1)y_1y_2}{(z_1-v_1)(z_2-v_2)-g_1g_2} \quad \ldots\ldots\ldots (28).$$

Since in this case the terms of the characteristic function which involve x_1 and x_2 are separated from those which involve y_1 and y_2, we may consider either apart from the other. In the case of an optical instrument symmetrical about its axis,

$$u_1 = v_1,\ u_2 = v_2,\ f_1 = g_1, \text{ and } f_2 = g_2 \quad \ldots\ldots\ldots\ldots (29).$$

Let l_1, l_2 be the tangents of the angles which the incident and emergent rays, projected on the plane of xz, make with the axis of z,

$$l_1 = \frac{\frac{dV}{dx_1}}{\frac{dV}{dz_1}} = \frac{(z_2-u_2)x_1 - f_2x_2}{(z_1-u_1)(z_2-u_2)-f_1f_2} \quad \ldots\ldots\ldots\ldots (30),$$

$$l_2 = \frac{(z_1-u_1)x_2 - f_1x_1}{(z_1-u_1)(z_2-u_2)-f_1f_2} \quad \ldots\ldots\ldots\ldots (31).$$

If the incident ray is parallel to the axis, $l_1 = 0$, and the equation of the emergent ray is $(z_2-u_2)x_1 - f_2x_2 = 0$ ..... (32).

The emergent ray cuts the plane of yz where

$$z_2 = u_2 \quad \ldots\ldots\ldots\ldots (33).$$

This therefore is the position of the *second principal focus.*

When $z_2 = u_2 + f_2,\ x_1 = x_2$ ..... (34),

or the ray is at the same distance from the plane of yz as before incidence. This gives the position of the *second principal plane.*

Its distance from the second principal focus is f_2, which is called the *second principal focal length.*

When $z_2 = u_2 + f_1$ and $x_2 = 0$, $l_1 = l_2$ ..... (35),

or every ray which passes through this point is equally inclined to the plane of yz before and after passing through the instrument. This point is called the *second focal centre.*

The distance of the emergent ray from the axis of z, when $z = z_2$, is given by the equation

$$f_2x_2 = (z_2-u_2)x_1 - l_1\{(z_1-u_1)(z_2-u_2)-f_1f_2\} \quad \ldots\ldots\ldots (36).$$

When $(z_1-u_1)(z_2-u_2)-f_1f_2 = 0$ ..... (37),

the term multiplied by l_1 vanishes. Hence all the rays which pass through the point (x_1, z_1) pass through (x_2, z_2), whatever their inclina-

tion to the axis. The points x_1, z_1 and x_2, z_2 are therefore conjugate foci, and x_2 is the image of x_1.

In this case $$\frac{x_2}{x_1} = \frac{z_2 - u_2}{f_2} = \frac{f_1}{z_1 - u_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (38);$$

or, in words,—*The distance of the object from the axis is to the distance of the image from the axis as the distance of the object from the first principal focus is to the first principal focal length, or as the second principal focal length is to the distance of the image from the second principal focus.*

Let h_1 be the distance at which an object of diameter x_2 must be placed from the eye that it may subtend an angle equal to that which it subtends when placed at z_2, and seen through the instrument by an

eye at z_1, $$h_1 = \frac{x_2}{l_1} \text{ when } x_1 = 0, \quad \text{or} \quad h_1 = \frac{\mu_1}{\dfrac{d^2 V}{dx_1\, dx_2}} \quad \ldots\ldots\ldots\ldots \quad (39),$$

or $$h_1 = \frac{1}{f_2}\{(z_1 - u_1)(z_2 - u_2) - f_1 f_2\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (40).$$

The quantity h_1 is that which occurs in Cotes' Theorem, and to which Smith gives the name of the "apparent distance."

Differentiating h_1 with respect to z_1 and z_2, we find

$$\frac{dh_1}{dz_1} = \frac{z_2 - u_2}{f_2}, \quad \frac{dh_1}{dz_2} = \frac{z_1 - u_1}{f_2}, \quad \frac{d^2 h_1}{dz_1 dz_2} = \frac{1}{f_2} \quad \ldots\ldots\ldots \quad (41).$$

When the focal length is infinite, the instrument becomes a *telescope*, and the characteristic function is

$$V = V_1 + \mu_1 z_1 + \mu_2 z_2 + \frac{1}{2}\frac{\mu_1^2 x_1^2 + m^2 \mu_2^2 x_2^2 + 2m\,\mu_1 \mu_2\, x_1 x_2}{\mu_1 (z_1 - u_1) - m^2 \mu_2 (z_2 - u_2)} + \text{a similar term in } y.$$

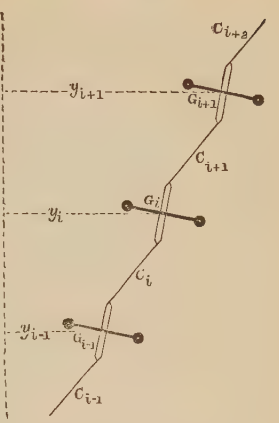
Here m is the angular magnification, and u_1, u_2 are the coordinates of any two conjugate foci. The linear magnification is $\dfrac{\mu_1}{\mu_2 m}$ and the elongation is $\dfrac{\mu_1}{\mu_2 m^2}$.

Vibrations and Waves in a stretched uniform Chain of Symmetrical Gyrostats. By Sir W. Thomson, F.R.S.

[*Read April* 8, 1875.]

A gyrostat is a rapidly rotating fly-wheel, frictionlessly pivoted on a stiff moveable framework or containing case. A symmetrical gyrostat is one in which not only the fly-wheel but the case is kinetically symmetrical round the axis of rotation of the wheel.

Let the chain consist of alternate gyrostats and massless connecting links, and let the connection be by universal flexure joints* at each end of each link. For simplicity, at present suppose the axes of the gyrostatic links to be all in one line when the chain is stretched straight. This line will be called the equilibrium axis.



Let g and c be the lengths of the gyrostatic and connecting links respectively; m the mass, and μ and ν the moments of inertia round the axis of figure and a line perpendicular to it through centre of inertia, of a gyrostatic link, fly-wheel and case included; let λ' be the moment of inertia of each fly-wheel alone, round its axis of figure; and let ω be the angular velocity once given to each fly-wheel, and remaining always the same because of the frictionlessness of the pivots. Instead of first investigating infinitesimal motions in general, we shall first take the particular case of circular motions not limited to being infinitesimal. Then, taking them to be infinitesimal; by composition of circular motions in similar or in contrary directions, and with different phases, we pass readily to the general solution of the problem of infinitesimal motions.

Problem.—Suppose a finite length of such a chain to be placed with its links forming an open plane polygon; and, the ends of the extreme links being held fixed by universal flexure joints, let each link be so set in motion perpendicularly to this plane that the polygon of axes moves as a rigid polygon rotating round the line joining the ends, with a given angular velocity n: required the form of the polygon, and the forces on the fixed ends, so that the chain, when left to itself, may continue revolving in the manner specified.

Call successive connecting and gyrostatic links .....C_i, G_i, C_{i+1}, G_{i+1}, ... Let ϑ_i and θ_i denote the inclinations of C_i and G_i to the line joining the ends, and let y_i be the distance of the centre of inertia of G_i from this line. We have the geometrical relation

$$y_{i+1}-y_i = \tfrac{1}{2}g\,(\sin\theta_{i+1}+\sin\theta_i)+c\,\sin\vartheta_{i+1} \quad \text{.....(1).}$$

Again, by the geometry of the universal flexure joint (Thomson and Tait, § 109), each gyrostatic link moves as if its axis were produced to the line joining the fixed ends, and there joined to a fixed object by a universal flexure joint. Hence the instantaneous axis of the motion of G_i bisects the angle $\pi-\theta_i$ between the line of its axis and the line

* Thomson and Tait's "Natural Philosophy," § 109.

joining the fixed ends. Its angular velocity round this instantaneous axis is $2n \sin \frac{1}{2}\theta_i$. The components of this round the axis of G_i, and a perpendicular to the axis of G through the equilibrium-axis, are

$$2n \sin^2 \tfrac{1}{2}\theta_i \text{ and } 2n \sin \tfrac{1}{2}\theta \cos \tfrac{1}{2}\theta_i, \text{ or } n(1-\cos\theta_i) \text{ and } n \sin\theta_i.$$

The corresponding moments of momentum are

$$(\lambda-\lambda').n(1-\cos\theta_i) \text{ and } \mu.n\sin\theta_i.$$

Hence the whole moment of momentum of G_i (case and fly-wheel) round the axis of G_i is

$$(\lambda-\lambda').n(1-\cos\theta_i)+\lambda'\omega.$$

Resolving this and $\mu n \sin\theta_i$, round the equilibrium axis and a perpendicular to it in the plane of the chain, we have, for whole component moment of momentum round the last mentioned line,

$$\{(\lambda-\lambda')n(1-\cos\theta_i)+\lambda'\omega\}\sin\theta_i-\mu n\sin\theta_i\cos\theta_i.$$

This line revolves with angular velocity n in a plane perpendicular to the equilibrium axis, and there must therefore be a couple equal to

$$n[\{(\lambda-\lambda')n(1-\cos\theta_i)+\lambda'\omega\}\sin\theta_i-\mu n\sin\theta_i\cos\theta_i],$$

acting on G_i round an axis perpendicular to the plane of the equilibrium axis and the axis of G_i. The direction of this couple, when positive, is such as to tend to diminish the angle θ_i. We are now ready to write down the equations of motion (or kinetic equilibrium) of G_i. The component parallel to the equilibrium axis, of the pull in the connecting links, must be the the same for all. (This *is* all the equations of motion parallel to the equilibrium line.) Let its amount be P: so that P sec ϑ_i is the pull in the connecting link C_i. The applied forces on G_i are the pulls of C_i and C_{i+1} on its ends. Resolving them we have:—

Parallel to equilibrium axis.	Perpendicular to equilibrium axis.
$-P$,	$-P\tan\vartheta_i$,
$+P$,	$+P\tan\vartheta_{i+1}$;

and, transposing to centre of inertia of G_i, we have finally

zero parallel to equilibrium axis;

$P(\tan\vartheta_{i+1}-\tan\vartheta_i)$ perpendicular to equilibrium axis;

and couple $-Pg\sin\theta_i+P(\tan\vartheta_{i+1}+\tan\vartheta_i).\frac{1}{2}g\cos\vartheta_i$

in plane through equilibrium axis and direction tending to increase θ_i. Hence, for motion of centre of inertia of G_i,

$$mn^2y_i+P(\tan\vartheta_{i+1}-\tan\vartheta_i)=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(2),$$

and, by couples,

$$n[\{(\lambda-\lambda')n(1-\cos\theta_i)+\lambda'\omega\}\sin\theta_i-\mu n\sin\theta_i\cos\theta_i]$$
$$=Pg\{-\sin\theta_i+\tfrac{1}{2}(\tan\vartheta_{i+1}+\tan\vartheta_i)\}\ldots\ldots(3).$$

Equations (1), (2), (3), applied to each gyrostatic link, give as many equations as there are of unknown quantities ϑ_i, θ_i, y_i, if we suppose one end of the chain to be a gyrostatic and the other a connecting link, so that there be the same number of the two kinds of links.

When the displacements and inclinations are infinitely small, the "algorithm of finite differences," as applied by Lagrange to the transverse oscillations of a "linear system of bodies" (a case of what the present problem becomes when $\omega = 0$) is conveniently applicable. Equations (1), (2), and (3), when we can neglect the cubes of θ_i and ϑ_i, become

$$y_{i+1} - y_i = \tfrac{1}{2} g\,(\theta_{i+1} + \theta_i) + c\vartheta_{i+1} \quad \ldots\ldots (4),$$

$$mn^2 y_i + \mathrm{P}\,(\vartheta_{i-1} - \vartheta_i) = 0 \quad \ldots\ldots (5),$$

$$n\,(\lambda'\omega - \mu n)\,\theta_i = \mathrm{P}g\,\{-\theta_i + \tfrac{1}{2}\,(\vartheta_{i+1} + \vartheta_i)\} \quad \ldots\ldots (6).$$

Let ϱ denote a symbol of operation such that

$$\varrho u_i = u_{i+1} \quad \ldots\ldots (7),$$

where u_i is any function of i.

From (6) we have

$$\theta_i = \frac{1}{2}\,\frac{\mathrm{P}g\,(\varrho+1)\,\vartheta_i}{n\,(\lambda'\omega - \mu n) + \mathrm{P}g}.$$

Using this in (4), and eliminating ϑ_i between (4) and (5), we find, finally,

$$\left[\mathrm{P}\,(\varrho-1)^2 + mn^2\left\{\frac{1}{4}\,\frac{\mathrm{P}g^2}{n\,(\lambda'\omega - \mu n) + \mathrm{P}g}\,(\varrho+1)^2 + c\varrho\right\}\right] y_i = 0 \ldots (8);$$

and the same holds, of course, with θ_i or ϑ_i substituted for y_i. Hence to determine ρ we have the quadratic

$$\rho^2 - 2\,(1-e)\,\rho + 1 = 0 \quad \ldots\ldots (9),$$

where

$$e = \frac{1}{2}\,\frac{\dfrac{\mathrm{P}g^2}{\mathrm{P}g + \lambda' n\omega^2 - \mu n^2} + c}{\dfrac{\mathrm{P}}{mn^2} + \dfrac{1}{4}\,\dfrac{\mathrm{P}g^2}{\mathrm{P}g + \lambda' n\omega^2 - \mu n^2}} \quad \ldots\ldots (10).$$

Put now

$$1 - e = \cos\alpha, \text{ or } \sqrt{\frac{e}{2}} = \sin\tfrac{1}{2}\alpha \quad \ldots\ldots (11).$$

The solution of (9) becomes

$$\rho = \cos\alpha \pm \sin\alpha\,\sqrt{-1},$$

and therefore the general solution of (8) is

$$y_i = \mathrm{A}\cos i\alpha + \mathrm{B}\sin i\alpha \quad \ldots\ldots (12);$$

and if x_i be the other coordinate of the centre of inertia of G_i (that of G_0 being taken as zero), we have

$$x_i = i\,(g+c) \quad \ldots\ldots (13).$$

Hence

$$y_i = \mathrm{A}\cos\frac{\alpha x_i}{g+c} + \mathrm{B}\sin\frac{\alpha x_i}{g+c} \quad \ldots\ldots (14);$$

that is to say, the centres of inertia of the links lie on a simple har-

monic curve, whose wave length is $\frac{2\pi}{\alpha}(g+c)$ .....(15).

The period is $\frac{2\pi}{n}$ .....(16);

and therefore, if V denote the velocity of propagation of the "circularly polarized" wave made up by proper superposition of our solutions, we have $V = \frac{n(g+c)}{\alpha}$ .....(17);

and therefore, by (11), $V = \frac{\sin\frac{1}{2}\alpha}{\frac{1}{2}\alpha}\frac{n(g+c)}{\sqrt{2e}}$ .....(18);

whence, by (10),

$$V = \frac{\sin\frac{1}{2}\alpha}{\frac{1}{2}\alpha}\left\{\frac{1+\frac{1}{4}\frac{mn^2g^2}{Pg+\lambda' n\omega-\mu n^2}}{1-\frac{g(\lambda' n\omega-\mu n^2)}{(g+c)(Pg+\lambda' n\omega-\mu n^2)}}\right\}^{\frac{1}{2}}\left\{\frac{P(g+c)}{m}\right\}^{\frac{1}{2}} \quad ...(19).$$

The first two factors of this expression become each equal to unity when α is infinitely small, that is to say, when the wave length (15) is infinitely great in comparison with the distance from centre to centre of neighbouring molecules, and the expression becomes simply

$$\left\{\frac{P(g+c)}{m}\right\}^{\frac{1}{2}} \quad ..... (20),$$

which is the known velocity of propagation of waves in a uniform stretched cord; $\frac{m}{g+c}$ being the mass per unit of length, and P the pull.

When α is not zero, but very small, we have approximately

$$\frac{\sin\frac{1}{2}\alpha}{\frac{1}{2}\alpha} = 1-\frac{(\frac{1}{2}\alpha)^2}{6} = 1-\frac{\pi^2}{6}\cdot\left(\frac{\alpha}{2\pi}\right)^2 = 1-\frac{\pi^2}{6}\left(\frac{g+c}{l}\right)^2;$$

where l denotes the wave length. And by the approximate expression (20) for V_1 or $\frac{nl}{2\pi}$, $mn^2 = 4\pi^2\frac{P(g+c)}{l^2}$ approximately. Also, because each link is very small in all its linear dimensions in comparison with l, $\frac{\mu}{ml^2}$ and $\frac{\lambda'}{ml^2}$ are each very small, and each comparable with $\frac{g^2}{l^2}$. Hence the second factor of (19) becomes approximately

$$\left\{\frac{1+\pi^2\frac{g(g+c)}{l^2}}{1-4\pi^2\frac{\mu}{ml^2}\left(\frac{\lambda'\omega}{\mu n}-1\right)}\right\}^{\frac{1}{2}} \text{ or } 1+\tfrac{1}{2}\pi^2\frac{g(g+c)+4\frac{\mu}{m}\left(\frac{\lambda'\omega}{\mu n}-1\right)}{l^2}.$$

And putting together the two factors, still approximately,

$$V = \left[1-\pi^2\left\{\frac{1}{6}\left(\frac{g+c}{l}\right)^2-\frac{1}{2}\frac{g(g+c)+4\frac{\mu}{m}\left(\frac{\lambda'\omega}{\mu n}-1\right)}{l^2}\right\}\right]\sqrt{\frac{P(g+c)}{m}}$$
..... (21).

The following presents were made to the Society in the Recess :—

"Journal of Institute of Actuaries," vol. xviii., Part 6, No. xcviii., Jan. 1875—No. xcix., April 1875.

"Annali di Matematica," tom. vii., fasc. 1°, Maggio 1875.

"Bulletin des Sciences Mathématiques et Physiques," Table des Matières et Noms d'Auteurs, tom. vii., 2e sémestre, 1874.

Ditto, tome huitième, Mars, Avril, Mai, Juni, 1875.

"Mémoires de la Société des Sciences Physiques et Naturelles de Bordeaux," tome x.

"Jahrbuch über die Fortschritte der Mathematik," (Ohrtmann, Müller,) fünfter Band, Heften 2 and 3, Berlin, 1875.

"Proceedings of Physical Society," Part ii., Nov. 7, 1874 to Jan. 18, 1875.

"Monatsbericht," März, April, Mai, Juni, 1875.

"Bulletin de la Société Mathématique de France," tome iii., Juillet N° 3, Août N° 4, Sept. N° 5, 1875.

"Crelle," achtzigster Band, 3es und 4es Heft.

"Journal de l'Ecole Polytechnique," quarante-quatrième cahier, tome xxvii., 1874.

"Smithsonian Report, 1873," Washington, 1874.

"Washington Astronomical and Meteorological Observations, 1872, made at the U. S. Naval Observatory," Rear-Admiral B. F. Sands, Washington, 1874.

"Memorie del R. Ist. Lombardo,"—Vol. XII., iii. della serie iii., fasc. vi. e ultimo; Vol. XIII., iv. della serie iii., fasc. i., Milano, 1874.

"Memorie del R. Ist. Lombardo," Rendiconti—Vol. V., fasc. xvii.–xx.; Vol. VI., fasc. i.–xx.; Vol. VII., fasc. i.–xvi.

"Transactions of Royal Irish Academy," Vol. xxv., Parts v.–ix., and Nos. x.–xix.

"Royal Irish Academy, Proceedings," Vol. i., series ii., Nos. 9, 10, Sessions 1872–3, 1873–4; Vol. ii., series ii., 1–3.

"Repertorium der Naturwissenschaften, monatliche Uebersicht der neuesten Arbeiten auf dem Gebiete der Naturwissenschaften," 1 Jahrgang, Nos. 1–6.

"Ueber die Verwerthung der elliptischen Functionen für die Geometrie der Curven dritten Grades," by Axel Harnack, Leipzig, 1875.

"Abstracts and Results of Magnetical and Meteorological Observations at the Magnetic Observatory, Toronto, from 1841–1871," Toronto, 1875.

"Ueber einen Beweis des Abel'schen Theoremes," by Axel Harnack, Erlangen, 12 Juli 1875.

"Le Valhalla des Sciences pures et appliquées," by the Count Leopold Hugo, Paris, 1875.

F. W. A. Argelander's "Bibliothek, Astronomie, Mathematik, Physik," Bonn.

"Zur Theorie der Ternären cubischen Formen," by Axel Harnack, Erlangen, März 1875.

"Zum Andenken an Otto Hesse," by Dr. Klein: from the Author, through J. J. Sylvester, F.R.S.

"Almindelige egenskaber ved Systemer af plane Kurver, med anvendelse til Bestemmelse af Karakteristikerne i de elementaere Systemer af fjerde Orden med 5 Tavler af H. G. Zeuthen," avec un résumé en français: from the Author. Kjøbenhavn, 1873.

APPENDIX.

By means of Prof. Sylvester's Fan (p. 78), it is possible to divide any angle into any assigned equal number of parts; and the trajectories of points taken in the several links connecting together the sticks of the fan have finite nodes, whose numbers are successively 1, 2^4, 3^4, 4^4, ...

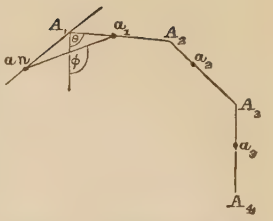
Prof. Sylvester stated, in his second communication (p. 78), That parallel motions exist at all is a paradox more wonderful than ever, now that his method gives the means of determining the conditions to be satisfied, and comparing their number with that of the disposable constants. The orders for 3, 5, 7, ... bars are 6, 20, 72, ... Formerly the existence of *one* was doubted; now a finite number for *every* order of linkwork is rendered highly probable. In particular, Prof. Sylvester shewed how to determine whether *Parallel Motions* exist, and, if so, how to find them for any given number of bars and mode of colligation. He shewed how to form a determinant involving only the lengths of the bars and other quantities which fix their direction; this determinant, if a parallel motion exists, must vanish identically for all values of the latter set of quantities. This is called the *Determinant of Parallel Motion*. The determinant is formed as a Jacobian of Equations, involving only linear functions of the lengths, and of a determinant corresponding to a set of equations of the same form as the above. Its evanescence gives a system of conditions to be satisfied, all expressed as rational functions of the lengths; and, by

known algebraical methods, these enable us to find *necessary* relations of the lengths, if a Parallel Motion exists. It must then be ascertained whether these solutions are sufficient, and the problem is solved.

Prof. Sylvester's remarks on "An Orthogonal Web of Jointed Rods" (p. 101) were to the following effect: If two sets of joints be taken respectively in two lines perpendicular to each other, either in a plane or in space, and a *linkage* be formed by connecting each point in one set with each point in the other by jointed rods, this constitutes an orthogonal web. It is *not* a fixture, and its motion is subject to this curious condition, that either each set of points must always continue to lie in the same right line, which may be called a neutral position, or else one set will lie in a right line, and the other in a plane at right angles to such line. Starting from the neutral position (a position of *double-lock*), the system may be said to be subject to an optional locking about one or the other of two perpendicular lines, and an unlocking about the others; but, when once put in motion, the system must again be brought into the same, or a new neutral position, before the one axis of lock can be got rid of, and another at right angles thereto substituted in its stead. If the whole motion be confined to a plane, the paradox consists in the link-combination forming one degree of liberty of deformation (αλλοίωσις, as distinguished by Plato from κίνησις), although a calculation of the amount of restraint by the general method applicable to such questions would seem to indicate that it ought to form an absolutely rigid system except in the case where there are only two joints in one at least of two sets. Taken in space, there is the further and more striking paradox, that the number of degrees of liberty of deformation, according to the choice made of one or the other of the two sets of points to be unlocked out of the rectilinear into the planar position, will be the *alternative of two numbers*, viz., the number of points in the one set or in the other set (which need not be the same), a kind of indeterminateness in the "Index of Freedom" without precedent in mathematical speculations. As lightning clears the air of impalpable noxious vapours, so an incisive paradox frees the human intelligence from the lethargic influence of latent and unsuspected assumptions. Paradox is the slayer of Prejudice.

We give here the Hydrostatical Problem discussed by Professor Wolstenholme (p. 114).

A tube of fine uniform bore is bent into the form of a regular polygon of n sides, and filled with equal volumes of n different fluids which do not mix; it is then closed, and held in any position in a vertical plane: the sides of the polygon formed by joining the common surfaces of the different fluids will always have constant directions; but, if two conditions be satisfied, every position will be one of equilibrium.



Let A_1, A_2, A_3, ... be the corners of the tube, a_1, a_2 ..... the common surfaces of the fluids, $A_1a_1 = A_2a_2 = \ldots\ldots = x$, θ the angle which A_1A_2 makes with the vertical, $\alpha = \frac{2\pi}{n}$ the angle between two adjacent sides; then, taking the length of a side as unity, we shall have for equilibrium, $\rho_1, \rho_2, \ldots\ldots \rho_n$ being the specific gravities of the different fluids,

$$x\{\rho_n\cos\theta+\rho_1\cos(\alpha-\theta)+\rho_2\cos(2\alpha-\theta)+\ldots+\rho_{n-1}\cos[(n-1)\alpha-\theta]\}$$
$$+(1-x)\{\rho_1\cos\theta+\rho_2\cos(\alpha-\theta)+\rho_3\cos(2\alpha-\theta)+\ldots\ldots$$
$$+\rho_n\cos[(n-1)\alpha-\theta]\}=0.$$

But, if ϕ be the angle which $a_n a_1$ makes with the vertical,

$$\frac{x}{1-x}=\frac{\sin(\alpha-\phi+\theta)}{\sin(\phi-\theta)};$$

therefore $x(\tan\phi\cos\theta-\sin\theta)=(1-x)(\sin[\alpha+\theta]-\tan\phi\cos[\alpha+\theta])$,

or $$\tan\phi=\frac{x\sin\theta+(1-x)\sin(\alpha+\theta)}{x\cos\theta+(1-x)\cos(\alpha+\theta)}$$

$$\frac{\sin\theta[\rho_1\cos\theta+\rho_2\cos(\alpha-\theta)+\ldots+\rho_n\cos(n-1\,\alpha-\theta)]-\sin(\alpha+\theta)[\rho_1\cos(\alpha-\theta)+\ldots+\rho_n\cos\theta]}{\cos\theta[\rho_1\cos\theta+\rho_2\cos(\alpha-\theta)+\ldots+\rho_n\cos(n-1\,\alpha-\theta)]-\cos(\alpha+\theta)[\rho_1\cos(\alpha-\theta)+\ldots+\rho_n\cos\theta]}$$

In the numerator the coefficient of ρ_r is

$$\sin\theta\cos[(r-1)\alpha-\theta]-\sin(\alpha+\theta)\cos r(\alpha-\theta)$$
$$=\tfrac{1}{2}[\sin(r-1)\alpha-\sin(r+1)\alpha]=-\sin\alpha\cos r\alpha,$$

and in the denominator is

$$\cos\theta\cos[(r-1)\alpha-\theta]-\cos(\alpha+\theta)\cos(r\alpha+\theta)$$
$$=\tfrac{1}{2}[\cos(r-1)\alpha-\cos(r+1)\alpha]=\sin\alpha\sin r\alpha;$$

therefore $$\tan\phi=-\frac{\rho_1\cos\alpha+\rho_2\cos2\alpha+\ldots+\rho_n}{\rho_1\sin\alpha+\rho_2\sin2\alpha+\ldots+\rho_{n-1}\sin(n-1)\alpha},$$

which is independent of θ. Hence any straight line joining two angular points of the regular polygon $a_1a_2\ldots a_n$ will be constant in direction, except when $\tan\phi$ takes the form $\frac{0}{0}$, which will be the case if

$$\rho_1\cos\alpha+\rho_2\cos2\alpha+\ldots\ldots+\rho_n\cos n\alpha=0,$$

and $$\rho_1\sin\alpha+\rho_2\sin2\alpha+\ldots\ldots+\rho_n\sin n\alpha=0.$$

That is, that if straight lines be drawn from a point parallel to the sides of the polygon and of lengths proportional to the densities of the fluids, these straight lines, considered as a system of forces, will balance.

If these conditions be not satisfied, the angle which the straight line joining the ends of any one of the fluids makes with the vertical, will

be found by drawing, as before, from a point straight lines parallel to the sides of a regular polygon of n sides, and of lengths proportional to the densities of the fluids taken in order; then, if that straight line which corresponds to the r^{th} fluid be vertical, the direction of the resultant of the n straight lines will be the direction of the straight line joining the ends of the r^{th} fluid.

In the case of $n = 3$, the conditions for every position being one of equilibrium can only be satisfied by $\rho_1 = \rho_2 = \rho_3$, which only gives an obvious result. If the densities be in arithmetical progression, the straight line joining the ends of the fluid of mean density will always be vertical. (It was my happening to notice this to be so in two particular cases which gave me the hint in general.)

If $n = 4$, the conditions for every position being one of equilibrium are $\rho_1 = \rho_3$, $\rho_2 = \rho_4$. That is, if we take equal volumes of two fluids which do not mix, each being equal to the content of two sides of the square, and divide each into two equal parts, and fill the tube by pouring the two fluids in alternately, closing the tube when filled, every position will be one of equilibrium.

If we take four fluids such that $\rho_1 + \rho_4 = \rho_2 + \rho_3$, $\tan \phi = 1$, so that the diagonals of the square formed by the surfaces of the fluids will be vertical and horizontal.

This instrument might be tried as a level and plumb line possibly; and perhaps some interesting toys might be made by other polygons.

It has been pointed out that the second part of the problem amounts to this, that the centre of gravity of the liquids is at the centre of the polygon: the two conditions assert that this is the case—for, if the liquids each fill a side of the polygon, the conditions are obvious; because $\Sigma(\rho, x) = 0$, and $\Sigma(\rho, y) = 0$, by simple projections,—and, if the liquids be displaced in any way, the centre of gravity of the several liquids will still form a regular polygon, and the same conditions will make the centre of gravity coincident with the centre of the polygon.

The Rev. C. Taylor's Paper, of which an Abstract is given pp. 123–125, has been published under the title "The Homographic Transformation of Angles," in "The Quarterly Journal of Pure and Applied Mathematics," No. 53, 1875.

At the June Meeting (p. 139) Professor Sylvester also described an apparatus for regulating the motion of a train of prisms, to which he has given the title of *isagoniostat*. In this place we call attention to an article by Dr. Sylvester "On the Plagiograph, *aliter* the Skew Pantigraph" ("Nature," Vol. 12, No. 296, July 1st, 1875). This article arose out of the study of linkages, and contains much interesting matter on the subject. The instrument is a simple modification of the Pantigraph, by means of which a figure, in the act of being magnified or reduced, may at the same time be slewed round the centre of similitude.

On the subject of Linkages we note "On a general method of producing exact Rectilinear Motion by Linkwork," by A. B. Kempe, B.A., in No. 163 (1875) of the "Proceedings of the Royal Society"; "On Three-bar Motion," by Professor W. Woolsey Johnson, "Messenger of Mathematics," No. lii. (August, 1875); "Sur les systèmes de tiges articulées," par M. V. Liguine, Prof. à l'Université d'Odessa, in the "Nouvelles Annales" (Dec. 1875, pp. 529–560). This last gives an account of work done by Sylvester, Peaucellier, Roberts, Hart, Kempe, and others.

In Mr. T. Cotterill's Paper (p. 139), S and T were taken homogeneous functions of any number of variables (say three, x, y, z); the degree of S being one lower than that of T, and are supposed to be connected with another set, x', y', z', of the same number of variables by the equations $\frac{x'}{x} = \frac{y'}{y} = \frac{z'S}{T}$. If the variables x, y, z, x', y', z' denote the coordinates of two points in a plane, a correspondence is established between them depending on the forms of S and T. The object of the communication was to explain the relations between the corresponding curves and to give examples.

Mr. Roberts informs us, with reference to his Paper "On the Rectification of the Arcs of Cartesians," Vol. V., pp. 6–9, that his attention has been kindly drawn by Professor Genocchi to a remarkable Paper of his, published in 1864, "Intorno alla rettificazione e alle proprieta delle caustiche secondarie," in which he has reduced the rectification of the arcs of Cartesians to the determination of three elliptic functions. This result was announced by him so early as 1855, but without demonstration, in a periodical of Turin, "Il Cimento." "The memoir," Mr. Roberts writes, "has not attracted the attention of mathematicians so much as it deserves; and as my principal result is clearly anticipated, I cannot do better than refer those interested in the subject to Professor Genocchi's valuable paper, published in the "Annali di Matematica pura ed applicata," Tomo VI.

Reference is made by Mr. N. M. Ferrers, M.A.,* in "Quarterly Journal of Mathematics, No. 53 (1875) to results obtained by Professor Cayley in Nos. 77, 78 of the Proceedings.

R. T.

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* "On the Potentials of Ellipsoids, Ellipsoidal Shells, Elliptic Laminæ, and Elliptic Rings, of variable Densities."

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† The reason for the delay in publication of this paper is given on p. 145 of Vol. V.

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